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A bounded complementary sensitivity function ensures topology-independent stability of homogeneous dynamical networks

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***Abstract*—This paper investigates the topology-independent stability of homogeneous dynamical networks, composed of interconnected equal systems. Precisely, dynamical systems with identical nominal transfer function $F(s)$ are associated with the nodes of a directed graph whose arcs account for their dynamic interactions, described by a common nominal transfer function $G(s)$. It is shown that topology-independent stability is guaranteed for all possible interconnections with interaction degree (defined as the maximum number of arcs leaving a node) equal at most to N if the ∞-norm of the complementary sensitivity function $NF(s)G(s)[1 + NF(s)G(s)]^{-1}$ is less than 1. This bound is non-conservative in that there exist graphs with interaction degree N that are unstable for an ∞-norm greater than 1. When nodes and arcs transferences are affected by uncertainties with norm bound $K > 0$, topology-independent stability is robustly ensured if the ∞-norm is less than $1/(1+2NK)$. For symmetric systems, stability is guaranteed for all topologies with interaction degree at most N if the Nyquist plot of $NF(s)G(s)$ does not intersect the real axis to the left of $-1/2$. The proposed results are applied to fluid networks and platoon formation.**

***Index Terms*—Dynamical networks, Directed graphs, Topology-independent stability, H_∞ norm.**

I. Introduction

Many complex technological and natural systems can be effectively modelled as a large-scale network of dynamic subsystems that interact with one another either statically or dynamically. Distributed problems in several contexts can be formulated in this dynamical-network framework [11], spanning from the analysis of biological and chemical systems [29], [10], [2], [19], [12] to IT systems [28], including consensus [23], [26], synchronisation [9], coordination [15], [8], estimation [27], [7] and control [14], [4], [5], [6], [21] of multi-agent systems.

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The robust stability of interconnected uncertain systems was analysed in [17], [16] for particular topologies, and in [1] resorting to integral quadratic constraints (IQC). Nyquist-like conditions that guarantee the stability of the entire network by satisfying local rules were derived in [20], [24] for heterogeneous interconnected systems; an IQC-based generalisation is in [18]. These Nyquist-like conditions scale with the network size and require at least a local knowledge of the network topology, in that they depend on the dynamics of the individual subsystem and their neighbours. This paper adopts a different viewpoint with respect to this body of literature. In fact, (1) *dynamic interconnections* are considered [22], [9] in addition to the individual subsystem dynamics, and (2) *topology-independent* stability conditions are sought, which certify stability regardless of the interconnection topology and of the size of the graph.

The dynamical networks considered here are represented by directed graphs, whose nodes correspond to nominally equal dynamic single-input single-output systems, each governed by a transfer function F, linked to one another according to a nominally common *dynamic* mechanism, represented by the transfer function G. Even if all the interconnected subsystems and their links evolve in a similar way, allowance is made for different (bounded) discrepancies between their actual transfer functions and their common nominal value, so that the proposed analysis is applicable to networks that are partly heterogeneous as well. No assumption is made on the network topology, except for the maximum number of arcs leaving a node, called interaction degree N. Then, given a pair of transfer functions F and G, do they ensure stability for any network topologies with a bounded interaction degree? The results of this paper address this question and can be summarised as follows.

- Given a maximum interaction degree N, the pair of stable transfer functions F, G ensures stability regardless of the interconnection topology if the complementary sensitivity function $NFG(1+NFG)^{-1}$ is stable and its ∞-norm is less than one.
- If the ∞-norm is greater than one, instability can occur for some graphs with interaction degree N.

- If (independent) disk-bounded uncertainties are present, the ∞-norm of the complementary sensitivity function must be bounded by $1/(1+2NK)$, where K is the uncertainty bound.
- For symmetric networks, where to each arc from node i to node j there corresponds an arc from node j to node i, the condition becomes less stringent, that is, the Nyquist plot of NFG must not intersect the real axis to the left of the point $-1/2$.

Numerical examples of a fluid network and a platooning problem illustrate the proposed results. Preliminary work in [3], based on the Nyquist plot, is limited to particular topology classes and common uncertainties.

II. Problem Statement

We consider a *directed graph* with n nodes and m arcs, whose structure is described by a (generalised) incidence matrix $B \in \{-1,0,1\}^{n\times m}$. Each column of B is associated with an arc and has an entry equal to -1 corresponding to the arc's departure node and an entry equal to 1 corresponding to the arc's arrival node. If some arc is directed to or comes from the external environment, the corresponding column has a single non-zero entry equal to -1 or 1, respectively.

The dynamic behaviour of the network is characterised by scalar variables $y_i(t)$, $i=1,\dots,n$, associated with the nodes (typically representing stored quantities) and scalar variables $u_h(t)$, $h=1,\dots,m$, associated with the arcs (typically representing flows). The variable characterising each node i is related to those characterising its incident arcs by means of a balance-like equation that, in terms of Laplace transforms, reads:

$$Y_i(s) = F(s)\sum_h B_{ih}\, U_h(s), \qquad (1)$$

where Y_i and U_h denote the Laplace transforms of y_i and u_h, respectively. In turn, each arc variable depends dynamically, yet linearly, on the variable y_i associated with its departure node. Precisely, if arc h is directed from node i to node j, in terms of Laplace transforms the following relation holds:

$$U_h(s) = G(s)Y_i(s). \qquad (2)$$

Note that, although this flow depends on y_i only, it affects both nodes i and j.

Remark 1: The case in which the net flow between node i and node j depends on *both* $y_i(t)$ and $y_j(t)$ can be accounted for by means of two distinct arcs, say h and k, directed in opposite directions, whose respective flows are given by $U_h(s) = G(s)Y_i(s)$ and $U_k(s) = G(s)Y_j(s)$.

Definition 1: A network is called *symmetric* if to any arc from node i to node j there corresponds an arc from node j to node i.

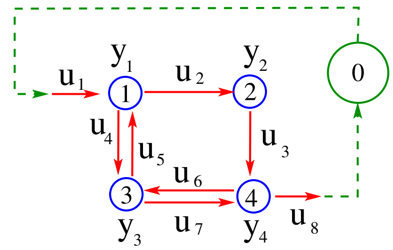

Figure 1. Network connected to the external environment (node 0).

Relations (1) and (2) apply also to arcs coming from/going to the external environment (node 0), with which a variable $y_0 \equiv 0$ is associated. However, no row in B is assigned to this node.

Let us now define matrix $\tilde{B}$ componentwise as

$$\tilde{B}_{ij} = \min\{0, B_{ij}\}.$$

Remark 2: Since the topology of a dynamical network is uniquely specified by its incidence matrix, B will be referred to in the following simply as the topology.

By introducing the vectors of node and arc variables:

$$\begin{aligned} Y(s) &= [\, Y_1(s) \;\; Y_2(s) \;\; \dots \;\; Y_n(s)\,]^\top, \\ U(s) &= [\, U_1(s) \;\; U_2(s) \;\; \dots \;\; U_m(s)\,]^\top, \end{aligned}$$

the system of equations for the arc-to-node transferences can be expressed in compact form as

$$Y(s) = F(s)B\,U(s)\,, \qquad (3)$$

and that for the node-to-arc transferences as

$$U(s) = -G(s)\,\tilde{B}^\top\, Y(s)\,. \qquad (4)$$

Therefore, the overall system's characteristic equation turns out to be

$$\det[I + F(s)G(s)\,B\tilde{B}^\top] = 0\,. \qquad (5)$$

Remark 3: $A = B\tilde{B}^\top$ has non-positive off-diagonal entries and positive diagonal entries. Moreover, it is column diagonally dominant, that is, $\sum_{i\neq j}|A_{ij}| \le A_{jj}$, $\forall j$. Therefore, $-A$ is a *compartmental matrix*. For symmetric networks, $B\tilde{B}^\top = L$, which is the so-called (symmetric) Laplacian matrix.

Example 1: The incidence matrix for the (asymmetric) network in Fig. 1 is

$$B = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \end{bmatrix},$$

matrix $\tilde{B}$ is

$$\tilde{B} = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and

$$A = B\tilde{B}^\top = \begin{bmatrix} 2 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

III. Topology-independent Stability with Bounded Interaction Degree

This section points out a necessary and sufficient condition that the network must fulfil to be stable in the sense specified by the following definition.

Definition 2: A dynamic graph characterized by the incidence matrix B, with all node transfer functions equal to F and all arc transfer functions equal to G, is *stable* if all the roots of the characteristic equation (5) are in the open left half-plane.

The next definition introduces a parameter, called interaction degree, that plays a crucial role in topology-independent stability analysis.

Definition 3: The *degree* of node i is the number of arcs leaving node i. It corresponds to the ith diagonal entry of matrix $B\tilde{B}^\top$ and is denoted by $\deg_i = [B\tilde{B}^\top]_{ii}$. The *interaction degree* is the largest of these values, that is, $\max_i \deg_i$.

To derive a topology-independent stability condition, given an upper bound N on the interaction degree, the following assumption is made.

Assumption 1: The transfer functions $F(s)$ and $G(s)$ are proper, asymptotically stable and with no purely imaginary zeros, and at least one of them is strictly proper (i.e., $F(\infty)G(\infty) = 0$).

The main result can now be stated.

Theorem 1: Functions $F(s)$ and $G(s)$ ensure the stability of system (3)–(4) for all topologies B with interaction degree equal at most to N if the complementary sensitivity function

$$W_N(s) \triangleq \frac{NF(s)G(s)}{1 + NF(s)G(s)} \tag{6}$$

corresponding to the loop transference $NF(s)G(s)$ is stable and its ∞–norm is less than 1, that is,

$$\sup_{Re(s)\geq 0} |W_N(s)| < 1\,. \tag{7}$$

Proof: Denote by $\mathcal{C}(c, r) = \{z \in C : |z - c| < r\}$ the open disk in the complex plane centred at c with radius r and by $\bar{\mathcal{C}}(c, r)$ the corresponding closed disk. Let $Q \in \mathbb{C}^{n\times n}$ be such that $B\tilde{B}^\top = Q\Theta Q^{-1}$ with Θ upper triangular. Then, the characteristic equation (5) can be written as

$$\begin{aligned} &\det[QQ^{-1} + F(s)G(s)Q\Theta Q^{-1}] \\ =\ &\det(Q[I + F(s)G(s)\Theta]Q^{-1}) \\ =\ &\det[I + F(s)G(s)\Theta] = 0 \end{aligned} \tag{8}$$

which must have no roots in the closed right half-plane. Since Θ is triangular, this requirement is satisfied if the equations $1 + F(s)G(s)\,\theta_i = 0$, with $\theta_i \in \sigma(\Theta) = \sigma(B\tilde{B}^\top)$, have no solutions in the closed right half-plane. Since $A = B\tilde{B}^\top$ is column diagonally dominant, according to Gershgorin circle theorem its eigenvalues lie in the union of the disks $\bar{\mathcal{C}}(\deg_i, \deg_i)$, $i = 1, \ldots, n$, which are all included in $\bar{\mathcal{C}}(N, N)$. It follows that

$$1 + F(s)G(s)\,\vartheta \neq 0 \text{ for } \mathrm{Re}(s) \geq 0,\ \vartheta \in \bar{\mathcal{C}}(N, N) \tag{9}$$

is a sufficient condition for stability. Now, based on properties of bilinear transformations, the following equivalence relations hold:

$$\left|\frac{Nz}{1 + Nz}\right| < 1 \Leftrightarrow \frac{1}{z} \notin \bar{\mathcal{C}}(-N, N) \Leftrightarrow z \notin \bar{\mathcal{C}}^{-1}(-N, N),$$

where $\bar{\mathcal{C}}^{-1}(-N, N) = \{\zeta : 1/\zeta \in \bar{\mathcal{C}}(-N, N)\} = \{\zeta : \mathrm{Re}(\zeta) \leq -1/(2N)\}$ (see Fig. 2). Therefore, if (7) is true, then $F(s)G(s) = -1/\vartheta$ cannot hold for $\mathrm{Re}(s) \geq 0$ and $\vartheta \in \bar{\mathcal{C}}(N, N)$, which implies (9). ■

The following result shows that bound (7) is not conservative.

Theorem 2: If

$$\sup_{Re(s)\geq 0} |W_N(s)| > 1\,, \tag{10}$$

there exists a topology B with interaction degree N such that system (3)–(4) is unstable.

Proof: In view of the maximum modulus theorem for holomorphic functions and the strict properness of $F(s)G(s)$, the supremum in (10) is indeed a maximum and is achieved on the imaginary axis. Therefore, (10) implies that $|W_N(\jmath\omega)| > 1$ for some $\omega > 0$. Equivalently, $|F^{-1}(\jmath\omega)G^{-1}(\jmath\omega) + N| < N$, which means that the Nyquist plot of $F^{-1}(\jmath\omega)G^{-1}(\jmath\omega)$ enters the disk $\mathcal{C}(-N, N)$, so that the Nyquist plot of $F(\jmath\omega)G(\jmath\omega)$ intersects the boundary of $\mathcal{C}^{-1}(-N, N)$ as in Fig. 2.

To complete the proof, we show that there is a sequence of graphs with interaction degree N, say $\{\mathcal{G}_k^N\}$, which correspond to matrices $B_k\tilde{B}_k^\top$ whose eigenvalues $(\theta_i)_k$ become dense on the boundary of $\mathcal{C}(N, N)$ as k increases. Consequently, the negative of their reciprocals $-1/(\theta_i)_k$ become dense on any bounded segment

$$\mathcal{S}_\mu = \{z : z = -1/(2N) + \jmath\beta, \quad |\beta| \leq \mu\},$$

lying on the boundary of $\mathcal{C}^{-1}(-N, N)$ (cf. Fig. 2), no matter how μ is chosen. Therefore, no matter how the Nyquist plot of FG intersects the boundary of $\mathcal{C}^{-1}(-N, N)$, it is always possible to choose a graph $\mathcal{G}_k^N$ and two eigenvalues θ_{kA} and θ_{kB} of the associated matrix $B_k\tilde{B}_k^\top$ such that the points $-1/\theta_{kA}$ and $-1/\theta_{kB}$ (belonging to $\mathcal{S}_\mu$) are encircled by the Nyquist plot a different number of times. As a consequence, either

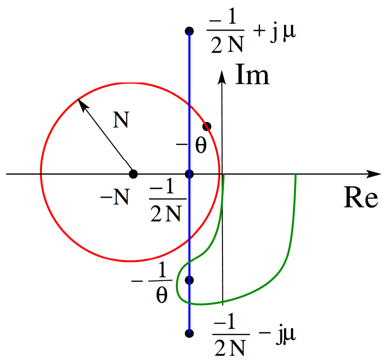

Figure 2. The disk $\mathcal{C}(-N, N)$ (region inside the red circumference) and its reciprocal set $\mathcal{C}^{-1}(-N, N)$ (region to the left of the vertical straight line passing through the blue segment).

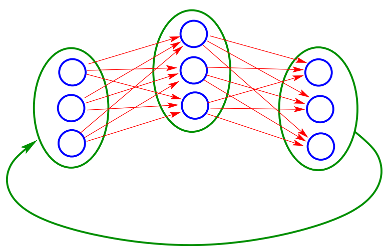

Figure 3. Block-circulant graph with 3-node clusters.

$1 + \theta_{kA}F(s)G(s) = 0$ or $1 + \theta_{kB}F(s)G(s) = 0$ exhibits at least one root with nonnegative real part. To construct the sequence of graphs, consider the block-circulant graph $\mathcal{G}_M^N$ consisting of M clusters of N nodes, where each node of a cluster is the departure node of N arcs connecting it to all of the nodes of the following cluster, thus forming a ring of subgraphs of the type depicted in Fig. 3 for $N = 3$. By indicating with Ω the $N \times N$ unit matrix ($\Omega_{ij} = 1$) and by I_N the $N \times N$ identity matrix, the $NM \times NM$ matrix $B\tilde{B}^\top$ for $\mathcal{G}_M^N$ turns out to be:

$$B\tilde{B}^\top = \begin{bmatrix} NI_N & 0 & \dots & 0 & -\Omega \\ -\Omega & NI_N & \dots & 0 & 0 \\ 0 & -\Omega & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\Omega & NI_N \end{bmatrix}.$$

The interaction degree of $\mathcal{G}_M^N$ is N. Since the symmetric matrix Ω can be diagonalised as $T^{-1}\Omega T = \hat{\Omega} = \text{diag}\{N, 0, \dots, 0\}$, its eigenvalues are 0 (with multiplicity $N-1$) and N. Now, by applying the transformation $\text{diag}\{T, T, \dots, T\}$ to matrix $B\tilde{B}^\top$, every block Ω is replaced by $\hat{\Omega}$ in the transformed matrix. By suitably grouping the rows and columns of $B\tilde{B}^\top$ we get

$$\hat{A} = \left[\begin{array}{cccccc|c} N & 0 & \dots & 0 & -N & & 0 \\ -N & N & \dots & 0 & 0 & & 0 \\ 0 & -N & \dots & 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -N & N & & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & & NI_p \end{array}\right],$$

where I_p is the identity of dimension $p = M(N-1)$. The characteristic polynomial for $\hat{A}$ is

$$p(\lambda) = \left[(\lambda - N)^M - (-N)^M\right](\lambda - N)^p$$

whose roots are N, with multiplicity p, and $\lambda_i = N(1 - r_i)$, $i = 0, \dots, M-1$, where $r_i = e^{\jmath\frac{2\pi i}{M}}$ are the Mth complex roots of unity. Therefore, as $M \to \infty$, the eigenvalues λ_i become dense on the boundary of the disk $\mathcal{C}(N, N)$. ■

Remark 4: Condition (7) is similar to condition (22) in [13] where, however, the connection matrix needs to be entirely known. Instead, Theorem 1 only requires the knowledge of the maximum number of arcs leaving a single node, i.e., the interaction degree. Therefore, the stability condition for two graphs with equal interaction degree is exactly the same, no matter how their nodes are connected.

Remark 5: For a fixed interaction degree N, the block-circulant network is the most prone to instability.

Remark 6: Condition (7) implies that the Nyquist plot of $F(\jmath\omega)G(\jmath\omega)$ must not cross the line $\text{Re}(s) = -\frac{1}{2N}$. This condition holds for arbitrary $N > 0$ if $\text{Re}[F(\jmath\omega)G(\jmath\omega)] \geq 0$, $\forall\omega$, which qualifies the product $F(s)G(s)$ as positive-real.

For symmetric networks (see Definition 1), $B\tilde{B}^\top$ is a symmetric matrix and, therefore, it has real eigenvalues. In this case the condition that $F^{-1}(\jmath\omega)G^{-1}(\jmath\omega)$ does not intersect $\mathcal{C}(-N, N)$ is replaced by the condition that $F^{-1}(\jmath\omega)G^{-1}(\jmath\omega)$ does not intersect the real segment $[-2N, 0]$. With reference to its reciprocal set, the following result holds.

Corollary 1: Functions $F(s)$ and $G(s)$ ensure the stability of system (3)–(4) for all symmetric networks with interaction degree equal at most to N if the Nyquist plot of $NF(s)G(s)$ does not cross the real axis to the left of $-1/2$: if $\text{Im}[NF(\jmath\omega)G(\jmath\omega)] = 0$, then $\text{Re}[NF(\jmath\omega)G(\jmath\omega)] > -1/2$.

IV. Robustness in the Presence of Independent Uncertainties

Even if a network is nominally homogeneous, in practice node and arc dynamics may be affected by independent uncertainties that may differ from one another. The following assumption will be adopted in this section.

Assumption 2: The uncertainties on the nominal transferences $F(s)$ and $G(s)$ are accounted for by diagonal matrices of stable transfer functions $\Delta_F(s)$ and $\Delta_G(s)$, whose diagonal entries $\Delta_{Fi}(s)$ and $\Delta_{Gi}(s)$ are subject to the relative bounds:

$$\sup_{\omega\in\mathbb{R}^+}\left|\frac{\Delta_{Fi}(\jmath\omega)}{F(\jmath\omega)}\right| < K_F,\ \sup_{\omega\in\mathbb{R}^+}\left|\frac{\Delta_{Gi}(\jmath\omega)}{G(\jmath\omega)}\right| < K_G. \quad (11)$$

Accordingly, in the presence of uncertainties the overall dynamical network is described by

$$Y(s) = [F(s)I + \Delta_F(s)]B\,U(s) \quad (12)$$
$$U(s) = -[G(s)I + \Delta_G(s)]\,\tilde{B}^\top Y(s) \quad (13)$$

The following result holds.

Theorem 3: The nominal functions $F(s)$ and $G(s)$ ensure the stability of system (12)–(13) for all topologies B with interaction degree at most N and uncertainty bound (11) if

$$\sup_{Re(s)\geq 0}\left|\frac{NF(s)G(s)}{1+NF(s)G(s)}\right| < \frac{1}{1+2K}, \quad (14)$$

where $K = K_G + K_F + K_G K_F$.

Proof: To ensure stability in the presence of uncertainties, the characteristic equation

$$\det[I + F(s)G(s)B\tilde{B}^\top + \Delta_F(s)BG(s)\tilde{B}^\top + F(s)B\Delta_G(s)\tilde{B}^\top + \Delta_F(s)B\Delta_G(s)\tilde{B}^\top] = 0$$

must not have roots in the closed right half-plane. To prove the thesis, resort is made to the zero-exclusion theorem. Since the nominal system is stable because condition (14) implies (7), the stability of the uncertain system is ensured if the previous determinant is nonzero at $s = \jmath\omega$ for all admissible Δ_F and Δ_G. Now, dividing by $F(\jmath\omega)G(\jmath\omega)$, this condition can be expressed as

$$\det[F^{-1}(\jmath\omega)G^{-1}(\jmath\omega)I + B\tilde{B}^\top + \Delta(\jmath\omega)] \neq 0 \quad (15)$$

where

$$\Delta(\jmath\omega) = \delta_F(\jmath\omega)B\tilde{B}^\top + B\delta_G(\jmath\omega)\tilde{B}^\top + \delta_F(\jmath\omega)B\delta_G(\jmath\omega)\tilde{B}^\top$$

and

$$\delta_F(\jmath\omega) = \frac{\Delta_F(\jmath\omega)}{F(\jmath\omega)} \quad \text{and} \quad \delta_G(\jmath\omega) = \frac{\Delta_G(\jmath\omega)}{G(\jmath\omega)}$$

are diagonal matrices whose entries are bounded according to (11). By omitting for simplicity the argument $\jmath\omega$, (15) can be rewritten as

$$\det[(F^{-1}G^{-1}+N)I + B\tilde{B}^\top - NI + \Delta] \neq 0 \quad (16)$$

which is satisfied if $(F^{-1}G^{-1}+N)$ is not an eigenvalue of the matrix $\Phi \triangleq NI - \Delta - B\tilde{B}^\top$. To proceed further, the following lemma, whose proof can be found in the Appendix, is useful.

Lemma 1: Matrix Φ is bounded in the 1-norm as

$$\|NI - \Delta - B\tilde{B}^\top\|_1 \leq N(1+2K)\,.$$

◇

Since the 1-norm is a bound for the eigenvalues, no eigenvalue of Φ lies outside the disk of radius $N(1+2K)$ centred at the origin. Therefore, stability is ensured if

$$|F^{-1}G^{-1} + N| \geq N(1+2K)$$

or, equivalently,

$$\left|\frac{NF(\jmath\omega)G(\jmath\omega)}{1+NF(\jmath\omega)G(\jmath\omega)}\right| \leq \frac{1}{1+2K},$$

which is ensured by (14). ■

V. Extensions: poles at origin and dual case

The previous results can be extended to the case of transfer functions $F(s)$ and $G(s)$ having, besides asymptotically stable poles, also poles in the origin. In this case, the fundamental requirement is that the graph is externally connected, which implies that $B\tilde{B}^\top$ is nonsingular (if, instead, $B\tilde{B}^\top$ had zero eigenvalues, there would be a cancellation at $s = 0$). Then condition (7) holds without changes. This allows us to deal, in particular, with buffer systems. Consider a flow network where the buffers at the nodes are integrators, so that $F(s) = 1/s$, and the controlled flows through the arcs have transfer function $G(s)$. Then, topology-independent stability with interaction degree at most N is ensured if

$$\left|\frac{NG(\jmath\omega)/\jmath\omega}{1+NG(\jmath\omega)/\jmath\omega}\right| < 1,$$

which is equivalent to the fact that Re $\left[\frac{G(\omega)}{\jmath\omega}\right] > -\frac{1}{2N}$. If the network is entirely symmetric, then this condition must be satisfied for all frequencies, if any, at which $G(\omega)/\jmath\omega$ intersect the real axis. If $G(s)$ is a first-order transfer function $G(s) = b/(s+a)$, $a, b > 0$, then an externally connected network is unconditionally stable under the assumption of symmetry, but it is not unconditionally stable otherwise. Similar considerations apply to synchronisation and consensus problems [23].

The conditions provided in the previous sections hold unchanged if we consider the dual matrix $[B\tilde{B}^\top]^\top = \tilde{B}B^\top$. Interestingly enough, the dual problem arises in contexts such as platooning [25], [26]. Assume that each node corresponds to an autonomous agent (vehicle, robot or craft) whose position is decided based on the position of other agents, with which it is possible to exchange information. The node equations in the Laplace domain can be written in this case as

$$Y_i(s) = F(s)\sum_{j\in\mathcal{N}_i} U_{ij}(s),$$

where $\mathcal{N}_i$ is the set indexing the agents that communicate with the ith agent, and $U_{ij}(s)$ is the correction imposed to agent i based on the difference $Y_i(s) - Y_j(s)$:

$$U_{ij}(s) = G(s)[Y_i(s) - Y_j(s)] \,.$$

The system characteristic equation is then

$$\det[I + G(s)F(s)\tilde{B}B^\top] = 0$$

and the analysis can be carried out using the previous arguments. The example in Section VI-B develops these considerations further.

Remark 7: In the dual case, the arc variables always depend on the difference $y_i - y_j$. The distinction between asymmetric and symmetric networks concerns the reciprocal influences. Precisely, in symmetric systems, to any input $U_{ij}(s) = G(s)[Y_i(s) - Y_j(s)]$ acting on node i there corresponds an input $U_{ji}(s) = G(s)[Y_j(s) - Y_i(s)]$ acting in the opposite direction on node j. This situation is briefly discussed in Section VI-B.

VI. Examples

A. A fluid network

Consider a network of reservoirs (nodes) and pipes (arcs), where node dynamics are described by the transfer function $F(s) = \rho/s$ (integrators) and arc dynamics account for the control law of the fluid flows. In particular, we assume that the flows among the reservoirs are controlled by PID controllers with transfer function

$$G(s) = \frac{K_D s^2 + K_P s + K_I}{s(1+\tau s)} \,. \tag{17}$$

The Nyquist plot of the open-loop function with $\rho = 1$, $K_P = K_I = K_D = 1$, $\tau = 1$ is shown in Fig. 4, which also shows the sets $\{z < -(2N)^{-1},\ z \text{ real}\}$ (thick black line) and $\{z : \Re\{z\} = -(2N)^{-1}\}$ (dashed red line) for the family of graphs with interaction degree $N = 2$. As in the case considered in Section V, two poles of the open-loop function are in the origin. The Nyquist plot enters the region to the right of the dashed red vertical line with abscissa $-1/(2N)$.

When the arc flows are functions of the departing node variables only (asymmetric case), the use of the PID controller (17) can be dangerous. Indeed, according to condition (7) of Theorem 1 and its consequences in terms of Nyquist diagram (Remark 6), such a choice can lead to instability. For instance, the characteristic equation (5) for the simple network depicted in Fig. 1 has the two unstable roots $0.0026 \pm 0.6611j$.

Instead, in the symmetric case, stability is ensured for any fluid network in view of Corollary 1, because the Nyquist plot does not intersect the thick black half-line on the real axis from $-1/(2N)$ to $-\infty$.

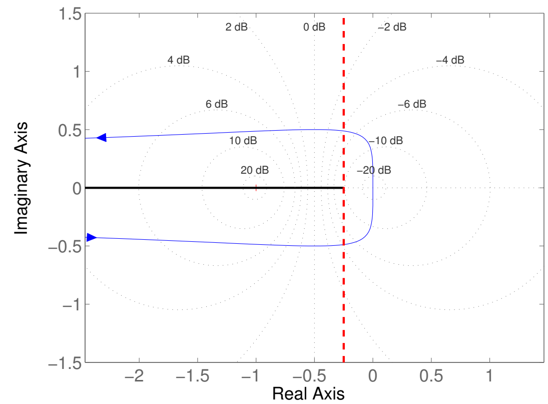

Figure 4. Nyquist diagram of the open-loop function for the example in Section VI-A

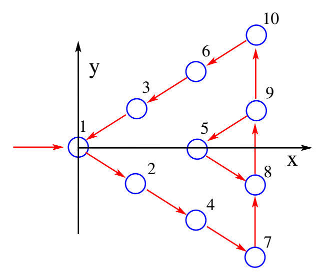

Figure 5. A possible configuration for a platooning problem.

B. A platooning problem

Consider now a platooning problem: several autonomous agents have to coordinate their positions to reach a particular configuration, for example the configuration represented in Fig. 5 along with the communication network. Suppose that the (decoupled) dynamics for the horizontal and vertical motion of the k-th agent are described by $X_k(s) = F(s)U_k(s)$ and $Y_k(s) = F(s)V_k(s)$, respectively, where

$$F(s) = \frac{1}{s(s+\alpha)} \,. \tag{18}$$

The control actions exerted on the nodes in the horizontal and vertical directions are

$$\begin{aligned} U_{ij}(s) &= G(s)[X_i(s) - X_j(s) - \bar{x}_{ij}/s] \,, \\ V_{ij}(s) &= G(s)[Y_i(s) - Y_j(s) - \bar{y}_{ij}/s] \,, \end{aligned}$$

where $\bar{x}_{ij}$ and $\bar{y}_{ij}$ are constant target values (step references) of the horizontal and vertical distances between agent i and agent j imposed by the target configuration.

This is the dual case described in Section V. We report next the results of simulations concerning a network of $n = 10$ agents, which need to be displaced so as to form equilateral triangles of unit edge (see Fig. 5) by means

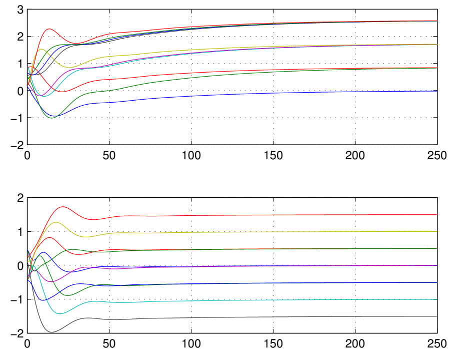

Figure 6. Time evolution of the agents' positions (top: x coordinate, bottom: y coordinate) in the asymmetric case for $\kappa = 0.25$.

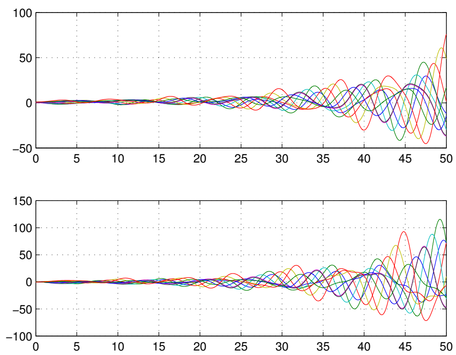

Figure 7. Time evolution of the agents' positions (top: x coordinate, bottom: y coordinate) in the asymmetric case for $\kappa = 2$.

of a static control action $G(s) = \kappa$. The parameter α in (18) has been set to 1.

The interaction degree of the network in Fig. 5 is clearly $N = 2$. The Nyquist plot of $\kappa/(s(s+1))$ does not intersect the vertical line through $-1/(2N)$ if $\kappa < 1/4 = 0.25$. According to Theorem 1 and Remark 6, this is a stability bound. For $\kappa = 0.25$, the system exhibits the stable transient shown in Fig. 6. For $\kappa > 0.25$, the network may become unstable, as shown by the plots in Fig. 7 corresponding to $\kappa = 2$.

In the symmetric case, it is assumed that to any force u_{ij} acting on node i, and depending on the difference of the positions of nodes i and j, there corresponds an equal and opposite force $u_{ji} = -u_{ij}$ acting on node j (see Remark 7). In a platooning context, this implies a pairwise coordination among agents, which confers more robustness to the system. Indeed, in this case the Nyquist plot does not intersect the negative real axis, hence Corollary 1 guarantees stability for any value of $\kappa > 0$. As Fig. 8 shows, the value $\kappa = 2$ does not give rise to instability.

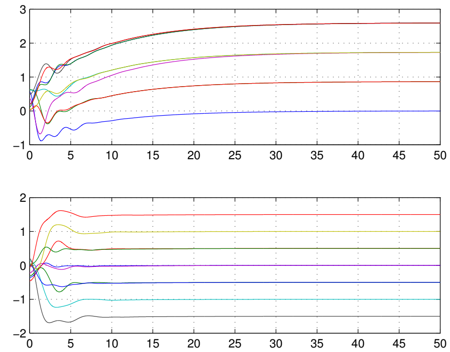

Figure 8. Time evolution of the agents' positions (top: x coordinate, bottom: y coordinate) in the symmetric case for $\kappa = 2$.

Note that, being the results independent of the network topology, they apply to different types of platoon configurations (e.g., square-shaped) and to platoons with an arbitrary number of members. Only the knowledge of the interaction degree N is required.

VII. Conclusions

It has been shown that topology-independent stability of a dynamical network with homogeneous node and arc dynamics is ensured if the complementary sensitivity function of the closed loop with loop gain depending on the interaction degree N has ∞-norm less than one. A less stringent condition holds in the symmetric case, whereas the asymmetric case is more critical. In the presence of disk-bounded uncertainties, a bound for the ∞-norm as a function of the uncertainty size has also been provided. The proposed results have been illustrated by means of numerical examples concerning a fluid network and a platoon formation problem.

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Appendix

Proof of Lemma 1. To prove the bound

$$\|\Phi\|_1 = \max_j \sum_i |\Phi_{ij}| \leq N(1+2K)\,,$$

we equivalently consider $\|-\Phi\|_1 = \|B\tilde{B}^\top - NI + \Delta\|_1$. Recall that matrix $B\tilde{B}^\top$ is column diagonally dominant with positive diagonal entries bounded by N. Therefore, if we subtract N from its diagonal entries, the sum of the absolute values of the column entries is bounded by N. Hence $\|B\tilde{B}^\top - NI\|_1 \leq N$. Consider now the matrix

$$\Delta = \delta_F B\tilde{B}^\top + B\delta_G\tilde{B}^\top + \delta_F B\delta_G\tilde{B}^\top\,.$$

We show that the 1-norm of the three terms appearing in the expression of Δ is bounded by $2NK_F$, $2NK_G$ and $2NK_GK_F$, respectively, so that $\|\Delta\|_1 \leq 2N(K_F + K_G + K_GK_F) = 2NK$. We prove the bound

$$\|\delta_F B\delta_G\tilde{B}^\top\|_1 \leq 2NK_GK_F$$

since the proofs of the other bounds are similar. Matrix Δ has the same 0 entries as $B\tilde{B}^\top$; hence it has at most N non-zero non-diagonal entries in each column. All the non-diagonal entries are minus the products of one entry of δ_G and one entry of δ_F. Therefore, their magnitude is bounded as

$$|\Delta_{ij}| \leq K_GK_F, \quad i \neq j\,.$$

The diagonal entries Δ_{ii} are products of the ith row of B and the ith column of $\tilde{B}^\top$. These entries are the sum of at most N products of entries δ_{G_i} and δ_{F_i}. Therefore

$$|\Delta_{ii}| \leq NK_GK_F\,.$$

Since there are at most N non-zero non-diagonal entries, the absolute values of the entries in each column sum up at most to $2NK_GK_F$. ■