# The Smallest Eigenvalue of the Generalized Laplacian Matrix, with Application to Network-Decentralized Estimation for Homogeneous Systems

Giulia Giordano, Franco Blanchini, Elisa Franco, Vahid Mardanlou, and Pier Luca Montessoro

**Abstract**—The problem of synthesizing network-decentralized observers arises when several agents, corresponding to the nodes of a network, exchange information about local measurements to asymptotically estimate their own state. The network topology is unknown to the nodes, which can rely on information about their neighboring nodes only. For homogeneous systems, composed of identical agents, we show that a network-decentralized observer can be designed by starting from local observers (typically, optimal filters) and then adapting the gain to ensure overall stability. The smallest eigenvalue of the so-called generalized Laplacian matrix is crucial: stability is guaranteed if the gain is greater than the inverse of this eigenvalue, which is strictly positive if the graph is externally connected. To deal with uncertain topologies, we characterize the worst-case smallest eigenvalue of the generalized Laplacian matrix for externally connected graphs, and we prove that the worst-case graph is a chain. This general result provides a bound for the observer gain that ensures robustness of the network-decentralized observer even under arbitrary, possibly switching, configurations, and in the presence of noise.

**Index Terms**—Graph Theory, Network problems, generalized Laplacian matrix, network-decentralized observer, network-decentralized estimation.

## 1 Introduction

Decentralized methods to monitor and control large-scale systems have become increasingly popular in the past decade [4]: centralized approaches are inefficient, especially in the presence of delays and limited communication bandwidth. Distributed problems related to multi-agent networks have been widely investigated [17], in particular concerning consensus [36], coordination [31], formation [20] and control [28].

Distributed estimation is an emerging problem. Distributed observer design has been considered in [21] for systems partitioned into disjoint areas; [47] proposes a consensus-based overlapping estimation framework; parallel estimation methods are proposed in [45] and hierarchical observer design strategies in [30]. LMI-based approaches are often useful to synthesize stable distributed observers [34], ensuring robustness to delays [6] and minimum-norm estimation error [49]. In the absence of a global coordinate system, formation problems benefit from estimating the states of neighboring agents: a distributed control method for robust global stabilization is proposed in [20], which improves the performance in the presence of delays in multi-hop communications [2]. An iterative gossip-based algorithm for common reference frame estimation in a bidimensional space is proposed in [23], while [56] uses a recursive state prediction based on local measurements and a collaboration unit providing gain updates to individual subsystems.

The network-decentralized control of naturally decoupled subsystems, performed by independent agents deciding their strategy based on restricted information [10], [11], [12], [13], [14], [19], is relevant in many applications [42], [48], such as congestion control [26], [27], dynamic routing in communication


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networks [28], [29], airplane and satellite formation [44], [54], vehicle platooning [39], [40] and power distribution networks [21]. When the subsystems cannot access the whole network information, due to security issues [23] or operational limitations [32], resorting to distributed estimators is inevitable and simplifies the control design [17].

Here we consider network-decentralized estimation (dual to network-decentralized control [10], [11], [12]): an observer is network-decentralized if its gain matrix has the same sparsity structure as the transpose of the overall output matrix. A similar problem is investigated in [22], where the sparsity structures of the observer gain and of the state matrix are matched within a distributed on-line Kalman filtering scheme. Our network-decentralized estimation problem consists in designing an observer where each agent must exchange information with the neighboring agents only, so as to reconstruct *its own state* (hence, we focus on *local* estimation, as opposed to the distributed global estimation problem, where each agent aims at estimating the state of the whole system; see [51] and the references therein) and it *has not even the knowledge* about the existence of the agents with which it cannot directly communicate. This problem is important in several contexts, including localization [1], [50] in multi-robot systems (for which decentralized and cooperative localization can be achieved by distributed Kalman filtering with sensor fusion [41] or minimum-entropy criteria [16]) and static sensor networks: [5] formulates the relative localization problem and proposes a synchronous algorithm based on Jacobi iterations; a distributed randomized algorithm is proposed in [38]; an asynchronous and distributed consensus-based algorithm for optimal localization based on noisy relative measurements is proposed in [18].

We focus on homogeneous systems, where all the agents in the network have identical dynamics. The agents are naturally decoupled and connected by the control action, and communicate according to an unknown and time-varying network topology. Building on [10], [11], we show that a necessary and sufficient condition for global convergence to zero of the network-decentralized estimation error is that the network is externally connected. Necessity is intuitive: for instance, to achieve localization, at least one of the communicating agents must know its absolute position. Sufficiency is proved constructively.

Our approach is based on the preliminary design of local observers; then, to achieve robust global stability, we take into account the network connectivity (poorly connected networks are more fragile than highly connected networks) via the *generalized* Laplacian matrix, which is *positive definite* for externally connected graphs. The second smallest eigenvalue of the *standard* Laplacian matrix (*i.e.*, the first positive eigenvalue, being the standard Laplacian positive semi-definite) is a measure of the graph connectivity [8], [24], [35], [40]; the smallest eigenvalue of the *grounded* Laplacian matrix, obtained by suitably removing rows and columns from the Laplacian, is also relevant in consensus problems with stubborn agents [37]. For an externally connected graph, we can consider the smallest eigenvalue of the generalized Laplacian, which is positive: we show that stability of the estimation scheme is robustly ensured if the observer gain is greater than a lower bound that depends on such an eigenvalue. To address the case in which the network topology is completely unknown and only the maximum number of nodes is available, we need to characterize the worst case. We prove a theorem that provides a fundamental support to our whole scheme, ensuring robustness against topology variations: precisely, we show that

(i) *for all externally connected networks with $N$ nodes, the smallest eigenvalue is lower bounded as $\lambda^*_{min}(N) \geq \frac{1}{\sigma_{max}(\Phi_N)}$, where $\sigma_{max}(\Phi_N)$ is the largest eigenvalue of the symmetric matrix $[\Phi_N]_{ij} = N + 1 - \max\{i, j\}$;*

(ii) *the bound is tight and is reached by the worst-case topology corresponding to a chain (a tree with a single leaf).*

The fact that the worst-case topology is a tree is expected, since trees are the internally connected graphs with minimal connectivity. This is also suggested in [53], although no proof is provided; our theorem, therefore, supports the claim and the results in [53].

In Section 2, we present and prove the theorem on the smallest eigenvalue of the generalized Laplacian matrix. In Section 3, we introduce the problem of network-decentralized estimation, focusing on homogeneous systems. Based on the theorem in Section 2, we provide a lower bound for the observer gain that guarantees robust stability of the estimation scheme in the presence of uncertain time-varying topologies (Section 4); for processes and measurements affected by disturbances, we provide an upper bound for the noise level that can be tolerated with a given observer gain (Section 4.1), which again relies on the theorem in Section 2. Finally, we provide application examples in Section 5: the first two examples (node localization and altitude detection) highlight that our scheme provides the optimal (least-square) solution in networks of integrators; a network of moving agents is also simulated, to provide further validation of the effectiveness of our estimation scheme.

## 2 The generalized Laplacian matrix and its smallest eigenvalue

In this section, we provide general results on the smallest eigenvalue of the generalized Laplacian matrix of a graph, which in Section 4 will be applied to the

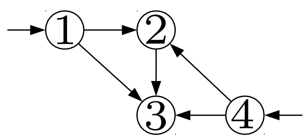


Fig. 1. Graph corresponding to matrix $G$ in Example 1.

network-decentralized observer problem for homogeneous systems, to give a lower bound for the gain that ensures stability (Theorem 4) even with unknown and switching topologies (Theorems 6 and 7), and an upper bound for the tolerable noise level in the case of noisy processes and measurements (Theorem 5).

Consider a directed graph formed by $N$ nodes connected by arcs, which can be either internal arcs, connecting two nodes, or external arcs, connecting one of the nodes with the external environment (a fictitious node $0$, not explicitly included in the graph). The graph is fully characterized by its generalized incidence matrix $G$, whose rows are associated with nodes and whose columns are associated with arcs:

- if arc $j$ connects node $k$ to node $h$, the $j$th column of $G$ has a $-1$ in the $k$th position and a $1$ in the $h$th position, and is zero elsewhere;
- if arc $j$ connects node $k$ with the external environment, the $j$th column of $G$ has a $1$ in the $k$th position, and is zero elsewhere.

***Example 1.*** The generalized incidence matrix

$$G = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

identifies a graph with four nodes, where internal arcs connect the pairs of nodes 1-2, 1-3, 2-3, 2-4 and 3-4, while external arcs connect node 1 and node 4 with the external environment (see Fig. 1).

Then, the generalized Laplacian matrix $\mathcal{L}$ is defined as

$$\mathcal{L} = GG^\top.$$

***Remark 1.*** A standard incidence matrix $G_0$ for the "extended" graph obtained by explicitly including also the fictitious node $0$, representing the external environment, is achieved by adding a row on the top of $G$, having a $-1$ in each column where $G$ has a single non-zero entry. In the case of Example 1,

$$G_0 = \left[\begin{array}{ccccccc} -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

is associated with the singular Laplacian matrix $G_0G_0^\top$, whose spectrum is $\{0, 2, 3, 4, 5\}$, while the spectrum of $GG^\top$ is $\{0.4384, 3, 4, 4.5616\}$.

The observations in the remark are useful to prove the next technical lemma.

***Lemma 1.*** The generalized incidence matrix $G$ is totally unimodular (*i.e.*, the determinant of each square sub-matrix achieved by selecting $k$ rows and $k$ columns of $G$ is either $0$, $-1$ or $1$).

*Proof:* The incidence matrix of a directed graph is always totally unimodular [43], [52]. Since matrix $G$ can be seen as a sub-matrix of an incidence matrix $G_0$, it is totally unimodular as well. $\square$

We introduce the following definitions.

***Definition 1.*** Two nodes of a graph are *adjacent* if they are connected by an arc in either direction. A *path* is a sequence of distinct nodes $i_1, i_2, \ldots, i_s$, where $i_k$ and $i_{k+1}$ (for $k = 1, \ldots, s-1$) are adjacent nodes. Nodes $i_1$ and $i_s$ are the *extrema* of the path.

***Definition 2.*** A graph is *internally connected* if each pair of nodes are the extrema of a path. A graph is *externally connected* if, for each node, a path exists connecting it to a node adjacent to node $0$ (namely, to the external environment). A graph is *connected* if it is both internally and externally connected.

***Remark 2.*** We consider non-oriented paths; hence, for instance, the graph in Fig. 1 is connected. Indeed, although the incidence matrix $G$ characterizes directed graphs, the direction of the arcs is no longer relevant when the Laplacian $\mathcal{L} = GG^\top$ is considered: if the sign of any column of $G$ is changed, $\mathcal{L}$ is the same. For this reason, in the previous definitions, the direction of the arcs is neglected.

We denote by $\lambda_{min}$ the smallest eigenvalue of the generalized Laplacian matrix $\mathcal{L} = GG^\top$ and by $\lambda^*_{min}(N)$ the smallest generalized Laplacian eigenvalue of all connected graphs with $N$ nodes. To characterize $\lambda^*_{min}(N)$, we need several preliminary results.

***Proposition 1.*** [7] The generalized Laplacian matrix $\mathcal{L} = GG^\top$ is non-singular (equivalently, $\lambda_{min} > 0$) if and only if the graph is externally connected.

The next proposition states that $\lambda_{min} > 0$ does not decrease if we augment the graph connectivity.

***Proposition 2.*** Consider two graphs represented by the generalized incidence matrices $G$ and $G'$. If the columns (arcs) of $G$ are a proper subset of those of $G'$, then

$$\lambda_{min}[GG^\top] \leq \lambda_{min}[G'G'^\top].$$

*Proof:* We can order the columns of $G'$ so that $G' = [G \ \Delta]$, where $\Delta$ includes all column vectors associated with the arcs in $G'$ that are not in $G$. Then

$$G'G'^\top = GG^\top + \Delta\Delta^\top,$$

with $GG^\top$ positive definite and $\Delta\Delta^\top$ positive semi-definite. Let $\lambda'_k$ be the ordered eigenvalues of $G'G'^\top$ and $\lambda_k$ those of $GG^\top$. Then, according to Weil's inequality [25], $\lambda'_k \geq \lambda_k \ \forall k$. In particular, this holds for the smallest eigenvalue. $\square$

Denote as *internal path* a path not including node 0 (the external environment). If a graph is not internally connected, it can be partitioned into *internally connected components*, each formed by a subset of nodes, such that:

- each component is internally connected;
- there is no internal path connecting two nodes $i$ and $j$ that belong to two distinct components.

Note that two internally connected components can be both connected to the external environment. Hence, a graph that is not internally connected is externally connected iff each of its internally connected components is externally connected.

If the graph can be partitioned into $C$ internally connected components, then, by suitably reordering the nodes, we can rewrite the generalized Laplacian in the block-diagonal form

$$GG^\top = \text{blockdiag}\{G_1G_1^\top, G_2G_2^\top, \ldots, G_CG_C^\top\},$$

where $G_jG_j^\top$ is the generalized Laplacian matrix associated with the $j$th internally connected component. Since the spectrum of $GG^\top$ is the union of the spectra of $G_jG_j^\top$, $j = 1, \ldots, C$, to characterize the smallest eigenvalue we can analyze, without restriction, the case of an internally connected graph (composed of a single internally connected component).

In graph theory, a *tree* with $N$ nodes is an internally connected graph having the smallest number $(N-1)$ of internal arcs (*i.e.*, an internally connected graph without cycles). We define a generalized tree as follows.

***Definition 3.*** A *generalized tree* is a tree with a single additional external arc, connecting one of the nodes to node 0. A generalized tree is a *chain* if it has a single branch (namely, each node is connected with at most two other nodes, including node 0).

Given all connected graphs (*i.e.*, internally connected graphs with at least one external connection) with $N$ nodes, the next result follows from Proposition 2.

***Proposition 3.*** Among all connected graphs with $N$ nodes, the graph corresponding to $\lambda^*_{min}(N)$ is a generalized tree.

*Proof:* Any other connected graph can be achieved by adding arcs to a generalized tree. In view of Proposition 2, adding arcs does not decrease the smallest eigenvalue. $\square$

Hence, $\lambda^*_{min}(N)$ is to be sought among all generalized trees. Our theorem proving that $\lambda^*_{min}(N)$ is achieved when the tree is a chain uses the following result in [46] (see also [55]).

***Proposition 4.*** [46] Let $\Phi_N$ be the matrix whose entries are $[\Phi_N]_{ij} = N + 1 - \max\{i, j\}$. Then, its largest eigenvalue is

$$\sigma_{max}(\Phi_N) = \left[2 + 2\cos\left(\frac{2\pi N}{2N+1}\right)\right]^{-1} \tag{1}$$

and can be approximated as $\sigma_{max}(\Phi_N) \approx 4N^2/\pi^2$.

Note that $\sigma_{max}(\Phi_N)$ is an increasing function of $N$.

***Theorem 1.*** Consider all possible generalized trees with $N$ nodes. Then,

$$\lambda^*_{min}(N) = \frac{1}{\sigma_{max}(\Phi_N)},$$

with $\sigma_{max}(\Phi_N)$ as in (1). Moreover, the tree corresponding to $\lambda^*_{min}(N)$ is a chain.

*Proof:* Let $G$ be the incidence matrix of any generalized tree with $N$ nodes. Since a generalized tree with $N$ nodes has $N$ arcs, $G$ is square. If we suitably order the nodes of the tree, matrix $G^\top$ is lower triangular:

$$G^\top = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \delta_{21} & 1 & 0 & \ldots & 0 \\ \delta_{31} & \delta_{32} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{N1} & \delta_{N2} & \delta_{N3} & \ldots & 1 \end{bmatrix},$$

with a single non-zero $\delta_{ij}$, equal to $-1$, in each row.

**Step 1**: $-G^\top$ is a Metzler matrix (*i.e.*, its off-diagonal entries are non-negative) and has negative-real-part eigenvalues. Hence $[G^\top]^{-1}$ is non-negative [9]. Moreover, $[G^\top]^{-1}$ is lower triangular, and its diagonal entries are all equal to 1.

**Step 2**: $G^\top$ is totally unimodular (cf. Lemma 1) along with its inverse. Therefore, any entry in the lower triangular part of $[G^\top]^{-1}$ is either 0 or 1.

**Step 3**: (Key point). Among all possible matrices $[G^\top]^{-1}$, that corresponding to the chain, $[\bar{G}^\top]^{-1}$, is the most populated by ones. In fact, $([\bar{G}^\top]^{-1})_{ij} = 1$ for all $i \geq j$:

$$[\bar{G}^\top]^{-1} = \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix}.$$

**Step 4**: Let $y = G^\top x$. Being $G^\top$ invertible (due to the external connection, see Proposition 1), $x = [G^\top]^{-1}y$. Then, $\lambda_{min}$ solves the optimization problem

$$\lambda_{min}^{1/2} = \inf_{x \neq 0} \frac{\|G^\top x\|}{\|x\|} = \inf_{y \neq 0} \frac{\|y\|}{\|[G^\top]^{-1}y\|} = \left[\sup_{y \neq 0} \frac{\|[G^\top]^{-1}y\|}{\|y\|}\right]^{-1}:$$

$\lambda_{min}^{1/2}$ is the inverse of the induced norm of $[G^\top]^{-1}$.

**Step 5**: Since the entries of $[G^\top]^{-1}$ are 1 on the diagonal and either 0 or 1 in the lower triangular part, the norm is maximized when *all* of these entries are 1, corresponding to $[\bar{G}^\top]^{-1}$ as in (2). Formally,

$$[G]^{-1}[G^\top]^{-1} \leq [\bar{G}]^{-1}[\bar{G}^\top]^{-1}$$

componentwise, since we are dealing with non-negative symmetric matrices; hence, the Frobenius eigenvalue of $[G]^{-1}[G^\top]^{-1}$ is maximized by $[\bar{G}]^{-1}[\bar{G}^\top]^{-1}$ [9]. The induced norm of $[G^\top]^{-1}$ is the square root of the Frobenius eigenvalue of $[G]^{-1}[G^\top]^{-1}$, which is maximized by taking $G = \bar{G}$, the incidence matrix of the chain.

Since $\Phi_N = [\bar{G}]^{-1}[\bar{G}^\top]^{-1}$, the proof is over. □

**The non-connected case.** Consider an externally connected graph with $N$ nodes, composed of $C$ internally connected components (each externally connected). By suitably reordering the nodes, its generalized Laplacian matrix can be written as $\mathcal{L} = \text{blockdiag}\{\mathcal{L}_1, \ldots, \mathcal{L}_C\}$, where $\mathcal{L}_i$ is the generalized Laplacian matrix of the $i$th internally connected component. Hence, denoting by $\lambda_{min}^{(i)}$ the smallest eigenvalue of $\mathcal{L}_i$, the smallest eigenvalue of $\mathcal{L}$ is $\lambda_{min} = \min\{\lambda_{min}^{(1)}, \ldots, \lambda_{min}^{(C)}\}$. The next corollary then immediately follows from Proposition 4 and Theorem 1.

***Corollary 1.*** Consider all possible externally connected graphs with $N$ nodes and $C$ internally connected components ($C \leq N$). Then,

$$\lambda_{min}^*(N) = \frac{1}{\sigma_{max}(\Phi_K)},$$

where $K$ is the number of nodes in the largest internally connected component ($K \leq N$) and $\sigma_{max}(\Phi_K)$ is defined as in (1). Moreover, the worst-case subgraph corresponding to the largest internally connected component is a chain.

Hence, in general, the worst case is a connected graph whose unique connected component is a chain.

## 3 Network-Decentralized Estimation

We consider $N$ agents with dynamics:

$$\dot{x}_i = A_i x_i + B_i u_i, \quad i = 1, ..., N,$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$. Agent $i$ is connected to its neighbors by a (non-empty) set of arcs $\mathcal{O}_i$. Measurements are associated with (undirected) arcs in the network of agents; for example the $j$th arc, connecting two agents $i$ and $k$, is associated with a relative measurement $y_j = (C_{ji}x_i + C_{jk}x_k)$, depending on the states of all of the nodes connected by the arc, where $C_{ij}$ are generic matrices (specific requirements will be needed later, see Assumption 1). Each agent runs a local estimator

$$\dot{z}_i = A_i z_i + B_i u_i + \sum_{j \in \mathcal{O}_i} L_{ij}(\hat{y}_j - y_j). \tag{3}$$

The estimated measurement $\hat{y}_j$ of each arc is:

$$\hat{y}_j = \sum_{k \in \mathcal{N}_j} C_{jk} z_k,$$

where $\mathcal{N}_j$ is the set that indexes the nodes connected by arc $j$. This is a general setup, valid even in the presence of hyper-arcs connecting more than two nodes.

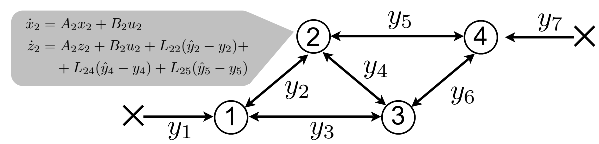


Fig. 2. The four-agent model in Example 2. Crosses indicate external connections (anchors).

***Example 2.*** Consider a system of four agents, associated with the nodes of the graph in Fig. 2. The agents exchange information about their state, with a communication topology given by the graph; measurement signals $y_i$ are associated with the arcs. Crosses indicate connections with the external environment (anchors): since nodes 1 and 4 are adjacent to node 0, $y_1$ depends exclusively on the state of agent 1, $y_7$ depends exclusively on the state of agent 4. Consider agent 1, with dynamics $\dot{x}_1 = A_1 x_1 + B_1 u_1$. Since it has knowledge about $y_1$, $y_2$ and $y_3$, its local observer is:

$$\begin{aligned}\dot{z}_1 = A_1 z_1 + B_1 u_1 &+ L_{11}(C_{11}z_1 - y_1)\\ &+ L_{12}(C_{21}z_1 + C_{22}z_2 - y_2)\\ &+ L_{13}(C_{31}z_1 + C_{33}z_3 - y_3).\end{aligned}$$

Agent 1 measures $y_2$ and $y_3$, receives the estimated outputs $C_{22}z_2$ from agent 2 and $C_{33}z_3$ from agent 3, and computes its estimated outputs: $C_{21}z_1$, transmitted to agent 2, and $C_{31}z_1$, transmitted to agent 3. Also, since node 1 is externally connected, agent 1 receives the "actual" output $y_1$ from the anchor and compares $y_1$ with its estimate $C_{11}z_1$.

The general dynamics of the overall system of the agents, along with their observer, is

$$\begin{cases}\dot{x} &= Ax + Bu,\\ y &= Cx,\\ \dot{z} &= Az + Bu - Ly + LCz,\end{cases} \tag{4}$$

where matrix $A \in \mathbb{R}^{n \times n}$ is a block-diagonal matrix, whose blocks $A_i$ are the individual agent matrices; matrix $C \in \mathbb{R}^{p \times n}$ has a block structure that depends on the communication topology of the agents. The dynamics of the estimation error $e = x - z$ is:

$$\dot{e} = Ae + L(y - Cz) = (A + LC)e. \tag{5}$$

***Example 3.*** For Example 2 (Fig. 2), we have

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ C_{31} & 0 & C_{33} & 0 \\ 0 & C_{42} & C_{43} & 0 \\ 0 & C_{52} & 0 & C_{54} \\ 0 & 0 & C_{63} & C_{64} \\ 0 & 0 & 0 & C_{74} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The overall estimator matrix is:

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & 0 & 0 & 0 & 0 \\ 0 & L_{22} & 0 & L_{24} & L_{25} & 0 & 0 \\ 0 & 0 & L_{33} & L_{34} & 0 & L_{36} & 0 \\ 0 & 0 & 0 & 0 & L_{45} & L_{46} & L_{47} \end{bmatrix},$$

which has the same block structure as $C^\top$.

***Definition 4.*** The observer defined by matrix $L$ is a *network-decentralized observer* if $L$ has the same block structure as $C^\top$.

***Definition 5.*** System (4) is *network-decentralized detectable* if there exists a network-decentralized observer such that the error dynamics (5) is asymptotically stable.

***Remark 3.*** The structure of $L$ corresponds to that of the incidence matrix $G$ of the interconnection graph. System (4) is externally connected iff at least one block-row in matrix $C$ has a single non-zero block.

In Example 2 (Fig. 2), nodes 1 and 4 are connected with the external environment: the first and the last row of matrix $C$ have a single non-zero block.

We denote as *unstable* an eigenvalue with a non-negative real part. A system is *detectable* if all of its unstable eigenvalues are observable. The next result follows, by duality, based on the results in [10].

***Theorem 2.*** If $A_i$ do not share unstable eigenvalues, then system (4) is network-decentralized detectable if and only if it is detectable.

When $A_i$ do not share unstable eigenvalues, a network-decentralized observer can be designed by dualizing the procedure in [10]. It is unclear, at present, whether the equivalence property in Theorem 2 holds, in general, in the presence of common unstable eigenvalues. However, interesting results can be found in special cases. In the sequel, we focus on homogeneous systems (where matrices $A_i$ are equal for all of the agents, each arc in the network connects at most two nodes and, if in a block-row of matrix $C$ there are two non-zero blocks, they are opposite), for which we provide an equivalence result for network-decentralized detectability in the following subsection (Theorem 3).

### 3.1 Network-decentralized detectability of homogeneous systems

We consider the following standing assumptions.

***Assumption 1.*** $A_i = A_1$ and $C_{ij} = \pm C_1$, for all non-zero blocks. Moreover there are at most two non-zero blocks $C_1$ for each block-row of $C$; if the non-zero blocks are two, they are opposite.

In the case of Example 2, we would have $C_{11} = C_1$, $C_{21} = -C_1$, $C_{22} = C_1$, $C_{31} = -C_1$, $C_{33} = C_1$, *etc.*

***Assumption 2.*** $(A_1, C_1)$ is detectable.

Assumption 2 is not restrictive, since the problem cannot be solved if this assumption is not verified.

The information exchange can be represented by a directed graph with incidence matrix $G$. In particular,

$$C = G^\top \otimes C_1,$$

where $\otimes$ denotes the Kronecker product (roughly, $C$ is the expansion of $G^\top$ achieved by replacing $\{-1, 0, 1\}$ entries with $\{-C_1, 0 \cdot C_1, C_1\}$ respectively). Note also that $A = I \otimes A_1$.

***Example 4.*** Consider four agents connected as in Fig. 2. Each agent $i$ aims at reconstructing its own position $r_i(t) \in \mathbb{R}^2$ in the plane (unknown to the agent itself) by exchanging information with its neighbors. Precisely, agent $i$ can update its own estimated position, $z_i(t)$, by communicating with all of its neighbors; all the non-zero blocks in $C$ are either $I_2$ or $-I_2$, where $I_2$ is the $2 \times 2$ identity matrix. If we assume that all of the agents are standing still, the equations are

$$\dot{r}_i(t) = 0, \quad \forall i,$$

hence $A_1 = \mathbf{0}_2$, the $2 \times 2$ zero matrix. Then the updating equation for agent 1 is

$$\dot{z}_1 = L_{11}(z_1 - \underbrace{r_1}_{y_1}) + L_{12}(z_1 - z_2 - \underbrace{(r_1 - r_2)}_{y_2}) + L_{13}(z_1 - z_3 - \underbrace{(r_1 - r_3)}_{y_3}).$$

The estimation error $e = r - z$ evolves as

$$\dot{e}(t) = LCe(t),$$

where $L$ must have the same block structure as $C^\top$.

***Remark 4.*** The graph needs not to be connected. We just require that each internally connected component is externally connected (its incidence matrix has at least one column with a single non-zero entry, corresponding to a single non-zero block in a block-row of $C$). Hence, each node of the graph is connected to the external environment by a suitable path.

***Lemma 2.*** System (4) is detectable if and only if there exists $\gamma \geq 0$ such that the Lyapunov inequality

$$A^\top P + PA - 2\gamma C^\top C < 0 \tag{6}$$

is satisfied for some $P > 0$. A stable observer is $L = -\gamma P^{-1} C^\top$.

*Proof:* It follows by duality from the results in [15, Section 7.2.1]. □

The observer $L = -\gamma P^{-1} C^\top$ is network-decentralized if $P$ is a block-diagonal matrix: for homogeneous systems, such a $P$ can always be found under proper conditions, as is constructively shown in the proof of the following theorem.

***Theorem 3.*** Under Assumptions 1 and 2, system (4) is network-decentralized detectable if and only if at least one of these conditions holds:

a) $A_1$ is asymptotically stable;
b) the system is externally connected.

*Proof:* **Sufficiency**

Condition a). If $A_1$ is asymptotically stable, then equation (6) holds for any $\gamma \geq 0$: by taking any $P_1 > 0$ such that $A_1^\top P_1 + P_1 A_1 < 0$ and then letting

$$P = \text{blockdiag}\{P_1, P_1, \ldots, P_1\}, \tag{7}$$

we get $A^\top P + PA < 0$ and the system is network-decentralized detectable, in view of Lemma 2 and of the block-diagonal structure of $P$.

Condition b). Consider the case of a connected graph (if the graph is composed of many internally connected components, the same reasoning can be applied to each of them). If $A_1$ is not asymptotically stable, we can choose $\gamma > 0$ sufficiently large to satisfy (6) for the single subsystem: $A_1^\top P_1 + P_1 A_1 - 2\gamma C_1^\top C_1 < 0$. By scaling $P_1$ (we replace $P_1$ with $\tilde{P}_1 = P_1/\gamma$ and keep on writing $P_1$ for simplicity), we obtain:

$$z_i^\top [A_1^\top P_1 + P_1 A_1 - 2C_1^\top C_1] z_i < 0, \quad \forall\, z_i \neq 0.$$

Network-decentralized detectability is ensured if we show that, for $\gamma > 0$ large enough, (6) is true with $P$ as in (7). Since $z^\top [C^\top C] z \geq 0$ in general, the assertion is proved if we show the following property [15]: for any non-zero $z \in \ker(C)$, we have

$$z^\top [A^\top P + PA] z < 0.$$

We can partition the state as $z = \begin{bmatrix} z_1^\top & z_2^\top & \ldots & z_N^\top \end{bmatrix}^\top$. In view of Assumption 1, for each block-row of $C$ with two non-zero blocks, say $h$ and $k$, we must have $C_1 z_h = C_1 z_k$. Since the graph is connected, we can put together the equalities corresponding to each block-row:

$$C_1 z_1 = C_1 z_2 = \cdots = C_1 z_N.$$

Moreover, due to the external connection, $C_1 z_l = 0$ for some $l$, since one block-row of $C$ has only the block $l$ different from zero. Hence, $C_1 z_k = 0$ for all $k$. Then, $z \in \ker(C)$ iff $z_i \in \ker(C_1)$, for all $i$, and thus

$$z^\top [A^\top P + PA] z = \sum_{i=1}^{N} z_i^\top [A_1^\top P_1 + P_1 A_1] z_i < 0,$$

in view of the detectability assumption on $(A_1, C_1)$. Therefore, for large $\gamma > 0$, we get (6).

**Necessity**

Assume by contradiction that neither condition a) nor condition b) holds. Let $\lambda$ be an unstable eigenvalue of $A_1$ and $z_1$ be an eigenvector of $A_1$ corresponding to $\lambda$. Then $z = [z_1^\top \; z_1^\top \ldots z_1^\top]^\top$ is an eigenvector of $A$. If the graph is not externally connected, then

$$Cz = 0,$$

since there are two opposite blocks in each block-row. Then

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} z = 0.$$

According to the Popov criterion, $\lambda$ is an unobservable eigenvalue. Being $\lambda$ unstable, the system is not detectable. □

From the proof of Theorem 3, the following corollary ensues.

***Corollary 2.*** Under Assumptions 1 and 2, system (4) is network-decentralized detectable if and only if it is detectable.

Theorem 3 does not provide any information on how large $\gamma > 0$ should be chosen. This problem is considered in the next section.

## 4 Computation of the Gain, Noise Rejection and Switching

In this section, we consider the case of an unknown, and potentially time-varying, topology. Inspired by the proof of Theorem 3, we adopt a design approach that requires solving the following two steps.

**STEP 1.** For the single subsystems $(A_1, C_1)$, design a gain $L_1 = -P_1^{-1} C_1^\top$, by means of some optimality criterion (*i.e.*, Kalman gain), where $P_1$ satisfies

$$A_1^\top P_1 + P_1 A_1 - 2C_1^\top C_1 < 0. \tag{8}$$

In this way, the system works even for isolated nodes.

**STEP 2.** Given a known matrix $P_1$ satisfying (8), to achieve global stability, apply the network-decentralized filter gain

$$L = -\gamma P^{-1} C^\top, \tag{9}$$

$$P = \text{blockdiag}\{P_1, P_1, \ldots, P_1\}, \tag{10}$$

for some $\gamma > 0$.

How large should $\gamma > 0$ be? If $A_1$ is asymptotically stable, then any $\gamma > 0$ is suitable. In the interesting unstable case, a lower bound for $\gamma$ is provided by the following theorem, in which the assumption of external connection is crucial.

***Theorem 4.*** Under Assumptions 1 and 2, let $G$ be the incidence matrix of the graph. If the graph is externally connected, then the error system with $L$ as in (9) is asymptotically stable provided that

$$\gamma > \gamma^* \doteq \frac{1}{\lambda_{min}[GG^\top]}, \tag{11}$$

where $\lambda_{min}[GG^\top]$ is the smallest eigenvalue of matrix $GG^\top \in \mathbb{R}^{N\times N}$.

*Proof:* Let us show that $A^\top P + PA - 2\gamma C^\top C < 0$ for $\gamma > \gamma^*$. Denoting by

$$\tilde{C}^\top \tilde{C} = \text{blockdiag}\{C_1^\top C_1, C_1^\top C_1, \ldots, C_1^\top C_1\},$$

this means

$$\underbrace{(A^\top P + PA - 2\tilde{C}^\top \tilde{C})}_{<0} + (2\tilde{C}^\top \tilde{C} - 2\gamma C^\top C) < 0.$$

The first addend in parentheses is negative definite, in view of its diagonal structure and of (8). Then, consider $(2\tilde{C}^\top \tilde{C} - 2\gamma C^\top C)$ and take $z = \begin{bmatrix} z_1^\top & z_2^\top & \ldots & z_N^\top \end{bmatrix}^\top$, $y_k = C_1 z_k$, $y = \begin{bmatrix} y_1^\top & y_2^\top & \ldots & y_N^\top \end{bmatrix}^\top$. The condition we look for is

$$\begin{aligned} & z^\top[2\tilde{C}^\top \tilde{C} - 2\gamma C^\top C]z = y^\top 2Iy - y^\top 2\gamma\Gamma\Gamma^\top y \\ & = y^\top 2[I - \gamma\Gamma\Gamma^\top]y < 0, \end{aligned} \tag{12}$$

where $\Gamma$ has the same structure as matrix $G$, once each 1 has been replaced by $I_p$ (the identity of dimension $p$) and each 0 by $\mathbf{0}_p$ (the zero $p \times p$ matrix); formally:

$$\Gamma = G \otimes I_p.$$

Hence, $\Gamma\Gamma^\top \in \mathbb{R}^{pN\times pN}$ has the same eigenvalues of $GG^\top \in \mathbb{R}^{N\times N}$, each repeated $p$ times. This means that (12) is true if (11) holds, and the proof is over. □

***Remark 5.*** The requirement (11) explains the necessity of assuming connection with the external environment: without external connections, we would have $\lambda_{min}[GG^\top] = 0$.

### 4.1 Noisy dynamics and measurements

To account for both process and measurement noise, we consider the model:

$$\begin{cases} \dot{x} &= Ax + Bu + Ev, \\ y &= Cx - w, \\ \dot{z} &= Az + Bu - Ly + LCz, \end{cases} \tag{13}$$

where $A$ and $C$ satisfy the previous Assumptions 1 and 2, while matrix $E \in \mathbb{R}^{n\times n}$ is block-diagonal, with blocks $E_1$. The disturbances $v$ and $w$ are assumed to be uncorrelated zero-mean stochastic processes with independent components,

$$\begin{aligned} &\mathcal{E}[v(t)] = 0, \quad \mathcal{E}[w(t)] = 0, \\ &\mathcal{E}[v(t)v(\tau)^\top] = \mu^2 I_n \delta(t-\tau), \\ &\mathcal{E}[w(t)w(\tau)^\top] = \mu^2 I_p \delta(t-\tau), \end{aligned} \tag{14}$$

where $0 < \mu \leq 1$ is a scaling factor. Both $v$ and $w$ are assumed to be uncorrelated with $x$. The dynamics of the estimation error $e = x - z$ are now:

$$\dot{e} = Ae + L(y - Cz) + Ev = (A + LC)e + Ev - Lw. \tag{15}$$

For each subsystem $(A_1, C_1)$ we can design an observer gain $L_1 = -P_1^{-1}C_1^\top$, so that

$$\dot{e}_1 = (A_1 + L_1C_1)e_1 + E_1v_1 - L_1w_1 \tag{16}$$

and, assuming $\mu = 1$, the performance for the local filter is bounded as (see, for instance, [33])

$$\lim_{t\to\infty} \text{trace}\mathcal{E}[e_1(t)e_1(t)^\top] \leq J_1 \doteq \text{trace}[P_1^{-1}],$$

where $P_1 > 0$ satisfies $P_1^{-1}(A_1 + L_1C_1)^\top + (A_1 + L_1C_1)P_1^{-1} + L_1L_1^\top + E_1E_1^\top \leq 0$, namely,

$$A_1^\top P_1 + P_1A_1 - C_1^\top C_1 + P_1E_1E_1^\top P_1 \leq 0. \tag{17}$$

The above are equalities if a local Kalman filter is considered. For $P_1$ that satisfies (17) as an equality, the performance is the $\mathcal{H}_2$ norm of the error system (16), with output taken as the state: $\mathcal{E}[e_1^\top(t)e_1(t)] = \text{trace}[P_1^{-1}] = J_1$. Based on Theorem 4, to ensure stability we need to apply $L = -\gamma P^{-1}C^\top$, where $P = \text{blockdiag}\{P_1, \ldots, P_1\}$, for some $\gamma > \gamma^*$. Yet, increasing $\gamma$ may cause performance degradation. To compare the overall filter performance with that of the individual observers, we look for the maximum noise level $\mu$ such that the global filter achieves the performance $J = NJ_1$, *e.g.*, the sum of the individual (optimal) performances. Indeed, with $L = L(\gamma)$, the global performance is bounded as

$$\lim_{t\to\infty} \sum_{k=1}^{N} \text{trace}\mathcal{E}\left[e_k(t)e_k(t)^\top\right] \leq$$

$$\text{trace}[\text{blockdiag}\{P_1^{-1}, \ldots, P_1^{-1}\}] = N\text{trace}\{P_1^{-1}\} = NJ_1,$$

with disturbances scaled by $\mu$ as in (14), provided that

$$P^{-1}(A+LC)^\top + (A+LC)P^{-1} + \mu^2 LL^\top + \mu^2 EE^\top \leq 0. \tag{18}$$

We seek the largest $\mu > 0$ compatible with (18), as a function of $\gamma$: its value exclusively depends on the graph topology and, again, on the smallest eigenvalue of the generalized Laplacian matrix.

***Theorem 5.*** Under Assumptions 1 and 2, let $G$ be the incidence matrix of the externally connected graph and let $\lambda_{min} = \lambda_{min}[GG^\top]$. System (13), with $L = -\gamma P^{-1} C^\top$, $P = \text{blockdiag}\{P_1, \ldots, P_1\}$ and $\gamma > \gamma^* = \frac{1}{\lambda_{min}}$, ensures the performance $NJ_1$ in the presence of measurement and process noises of magnitude $\mu$ as in (14), provided that

$$\mu < \mu^* = \sqrt{\frac{2\gamma\lambda_{min} - 1}{\gamma^2 \lambda_{min}}}. \tag{19}$$

If $\gamma = \gamma^* = \frac{1}{\lambda_{min}}$, then the worst-case maximum noise level is

$$\mu^* = \sqrt{\lambda_{min}} \leq \sqrt{\frac{2\gamma\lambda_{min} - 1}{\gamma^2 \lambda_{min}}}. \tag{20}$$

Moreover, for a given $\mu \leq \sqrt{\lambda_{min}}$, the performance $NJ_1$ is guaranteed for

$$\frac{\lambda_{min} - \sqrt{\lambda_{min}^2 - \lambda_{min}\mu^2}}{\lambda_{min}\mu^2} < \gamma < \frac{\lambda_{min} + \sqrt{\lambda_{min}^2 - \lambda_{min}\mu^2}}{\lambda_{min}\mu^2}. \tag{21}$$

*Proof:* For $L = -\gamma P^{-1} C^\top$, $P > 0$, condition (18) is equivalent to

$$PA + A^\top P + (\mu^2\gamma^2 - 2\gamma) C^\top C + \mu^2 P E E^\top P \leq 0.$$

We also have $PA + A^\top P - \tilde{C}^\top \tilde{C} + PEE^\top P \leq 0$ ($\tilde{C}^\top \tilde{C}$ is defined in the proof of Thm. 4), in view of (17) and of the block-diagonal structure. Hence, it is enough to show that

$$(\mu^2 - 1) PEE^\top P + [(\mu^2\gamma^2 - 2\gamma) C^\top C + \tilde{C}^\top \tilde{C}] \leq 0.$$

The first addend is negative semi-definite, due to our assumptions on $\mu$. As for the second addend, take $z$ and $y$ as in the proof of Theorem 4, and $\Gamma = G \otimes I_p$. We want $z^\top [(\mu^2\gamma^2 - 2\gamma) C^\top C + \tilde{C}^\top \tilde{C}] z = y^\top (\mu^2\gamma^2 - 2\gamma) \Gamma\Gamma^\top y - y^\top I y \leq 0$. This amounts to requiring $(\mu^2\gamma^2 - 2\gamma)\Gamma\Gamma^\top + I \leq 0$, that is, $(\mu^2\gamma^2 - 2\gamma)\lambda_i + 1 \leq 0$ for all $\lambda_i$ eigenvalues of $GG^\top$ (since $\Gamma\Gamma^\top$ has the same eigenvalues of $GG^\top$). Then, for a given $\gamma > \gamma^*$, for all $\lambda_i \in \sigma(GG^\top)$ it must be

$$\mu^2 \leq \frac{2\gamma\lambda_i - 1}{\gamma^2 \lambda_i} = f(\gamma, \lambda_i).$$

Since $\gamma > \frac{1}{\lambda_{min}}$, $\gamma\lambda_i > 1$ for all $i$. Under this assumption, $f$ is a decreasing function of $\gamma$, while it is always an increasing function of $\lambda_i$. Hence, the maximum noise level is $\mu^* = \sqrt{\frac{2\gamma\lambda_{min} - 1}{\gamma^2 \lambda_{min}}}$. Replacing $\gamma = \frac{1}{\lambda_{min}}$ in (19) immediately provides (20). Moreover, for a given $\mu$, we must have $\mu^2 \lambda_i \gamma^2 - 2\lambda_i \gamma + 1 \leq 0$ for all $\lambda_i \in \sigma(GG^\top)$; this is ensured if the inequality holds for $\lambda_i = \lambda_{min}$, which is true if $\gamma$ satisfies (21). □

***Remark 6.*** Given a graph where each node has at most one external connection, consider the standard incidence matrix $H$ and the diagonal matrix $K \in \mathbb{Z}^{N \times N}$, with $K_{ii} = 1$ if node $i$ is externally connected and $K_{ii} = 0$ otherwise. Then, the generalized Laplacian is given by $GG^\top = K + HH^\top$. In particular, when all of the nodes have one external connection, $K$ is the identity matrix and $\lambda_{min} = 1$. Hence, the worst-case bound (20) consistently gives $\mu^* = 1$, ensuring the ideal performance.

### 4.2 Unknown and switching topologies

To ensure stability even when the network topology is unknown and switching, a robust bound on $\gamma^*$ must be determined. Henceforth we assume that the incidence matrix, which fully characterizes the network topology, depends on time: $G = G_t$.

***Assumption 3.*** The incidence matrix $G_t \in \mathbb{R}^{N \times m_t}$, where $N$ is the number of agents (nodes) and $m_t$ is the number of arcs, belongs to a given family $\mathcal{G}(N, n_A)$, where $n_A$ is the number of anchors (*i.e.*, connections with the external environment). The current topology $G_t$ is unknown to the agents.

Given $N$ and $n_A$, we seek a robust bound on $\gamma^* \geq 0$ such that, if $\gamma > \gamma^*$, the network-decentralized observer remains stable under arbitrary switching $G_t \in \mathcal{G}(N, n_A)$.

To this aim, requiring that all the matrices in $\mathcal{G}(N, n_A)$ have full rank is necessary due to the nature of the problem, as discussed next. Any graph can be uniquely partitioned into internally connected components, so that, by means of a proper node and arc ordering, its incidence matrix $G$ can be written as

$$G = \text{blockdiag}\{G_1, \ldots, G_k\}. \tag{22}$$

***Proposition 5.*** The incidence matrix $G$ has full row rank if and only if each internally connected component of the graph is externally connected (namely, each block-matrix in (22) has at least one column with a single non-zero entry).

*Proof:* Matrix $G$ has full row rank if and only if there is no row vector $z = [z_1 \; z_2 \; \ldots z_k] \neq 0$ such that $zG = 0$ (*i.e.*, $zG = 0$ implies $z_i = 0$ for all $i$). If each internally connected component is externally connected, then, for all $i$, $G_i$ has full row rank, hence $z_i G_i = 0$ implies $z_i = 0$. Conversely, if one internally connected component is not externally connected, then $\bar{1}^\top G_i = 0$ (where $\bar{1}^\top = [1 \; 1 \ldots 1]$) because each column of $G_i$ has two non-zero entries equal, respectively, to 1 and to $-1$. □

If the graph has several internally connected components, each of them must be externally connected; otherwise, it may be subject to a "drift". Hence, the rank requirement is intrinsically necessary.

Real switching topologies (*e.g.*, wireless sensor networks with unreliable communications due to interference, packet losses or delays) may not meet the rank requirement. Then, convergence can still be ensured for the components that remain externally connected even under switching; for isolated components, convergence is possible up to a constant vector with all equal node components: $\bar{v} = \bar{1}^\top \otimes v$. For instance, in a localization problem, nodes belonging to an isolated internally connected component can determine their relative position only.

***Theorem 6.*** Assume that $A_1$ is not asymptotically stable. Then the following statements are equivalent.

(i) All the matrices in $\mathcal{G}(N, n_A)$ have full rank $N$.

(ii) There exists $\gamma^* \geq 0$ such that, for $\gamma > \gamma^*$, the network-decentralized observer is stable under arbitrary switching $G_t \in \mathcal{G}(N, n_A)$.

*Proof:* (i) $\Rightarrow$ (ii) follows from Theorem 4. Indeed, if all the incidence matrices $G \in \mathcal{G}(N, n_A)$ have full row rank, all the corresponding Laplacian matrices $GG^\top$ have full row rank as well. Then, we can take

$$\gamma^* \doteq \min_{G \in \mathcal{G}(N,n_A)} \frac{1}{\lambda_{min}[GG^\top]} \tag{23}$$

(note that there are finitely many $G \in \mathcal{G}(N, n_A)$) and see, with the same machinery of the proof of Theorem 4, that $P$ as in (10) provides a common quadratic Lyapunov function.

(ii) $\Rightarrow$ (i). Assume by contradiction that some $\tilde{G} \in \mathcal{G}(N, n_A)$ does not have full row rank. Then, if $G_t = \tilde{G}$, we do not have asymptotic stability and there exists at least one internally connected component of the graph that is not externally connected. Since by assumption $A_1$ is not asymptotically stable, let us take a (column) eigenvector $z_1$ associated with an unstable eigenvalue $\lambda$: $A_1 z_1 = \lambda z_1$, with $\mathrm{Re}(\lambda) \geq 0$. Assume that the internally connected component that is not externally connected is the first one, having $N_1$ nodes. Let $A$ be partitioned as $A = \text{blockdiag}\{\bar{A}_1, \bar{A}_2\}$, where $\bar{A}_1$ includes the $N_1$ blocks associated with the internally connected component that is not externally connected. Consider the non-zero vector $z^\top = [z_1^\top \; z_1^\top \ldots z_1^\top \; 0 \; 0 \ldots 0] = [\bar{z}_1^\top \;\; \bar{0}^\top]$, selecting the first subsystem. Then, by proceeding exactly as in Theorem 3, we have

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} z = \begin{bmatrix} \lambda I - \bar{A}_1 & 0 \\ 0 & \lambda I - \bar{A}_2 \\ \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{0} \end{bmatrix} = 0.$$

Therefore the system has an unobservable unstable eigenvalue, and the proof is over. □

The previous result has a drawback: we need to identify all possible topologies in the set $\mathcal{G}(N, n_A)$. However, we can provide a robust bound based on the exclusive knowledge of the number of nodes.

***Theorem 7.*** Let $G_t$ be the incidence matrix of any connected graph with $N$ nodes. Then, stability is ensured if

$$\gamma > \gamma^* \doteq \frac{1}{\lambda^*_{min}(N)} = \sigma_{max}(\Phi_N). \tag{24}$$

*Proof:* It follows from Theorems 6 and 1. □

***Corollary 3.*** Let $G_t$ be the incidence matrix of any connected graph with $N \leq \bar{N}$ nodes. Then, stability is ensured if $\gamma > 1/\lambda^*_{min}(\bar{N}) = \sigma_{max}(\Phi_{\bar{N}})$.

*Proof:* It follows from the fact that $\sigma_{max}(\Phi_N)$ is an increasing function of $N$, as can be seen from its expression (1) in Proposition 4. □

Finally, there is an interesting case in which we can show that any $\gamma > 0$ is suitable.

***Proposition 6.*** Assume that the system is driftless, namely $A = 0$, and that $C_1$ has full column rank. Then, if condition (i) of Theorem 6 is satisfied, the network-decentralized observer with $P = I$ is stable for any $\gamma > 0$.

*Proof:* Since $G_t$ has full row rank, $C_t = G_t^\top \otimes C_1$ has full column rank due to our assumption on $C_1$. The observer error equation is $\dot{e} = -\gamma C_t^\top C_t e$, with $C_t^\top C_t$ positive definite for all $t$. Therefore, all matrices $-\gamma C_t^\top C_t$ share the common quadratic Lyapunov function $V(e) = e^\top e$. □

The assumption on $C_1$ in Proposition 6 is not restrictive: if $C_1$ did not have full column rank, then $(A_1, C_1)$ would not be detectable, against Assumption 2.

# 5 Applications

## 5.1 Localization of nodes

Consider a set of agents, each willing to establish its own position $r_i$ based on some absolute information (*e.g.*, data from GPS), if available, and by exchanging information with neighboring agents, as in Example 4. For instance, agents in an unknown region may exchange information to construct a topographic map. Two communicating agents measure their distance and the angle formed by the segment between them and an absolute reference direction. Clearly, some of the agents must know their absolute position, hence they must be connected to external anchors.

We can straightforwardly apply the theory and take $P = I$, so that the observer gain is $L = -\gamma C^\top$ and the error system $\dot{e} = -\gamma C^\top C e$ is stable as long as the graph is connected.

To highlight an interesting feature of the proposed strategy, consider the effect of additive noise: $y = Cx - \delta$. The error equation associated with the resulting network-decentralized observer is $\dot{e} = -\gamma C^\top C e + \gamma C^\top \delta$ and, asymptotically, we have

$$e(\infty) = (C^\top C)^{-1} C^\top \delta,$$

which is the least-square solution of

$$\min \|Ce - \delta\| = \min \|Cz - y\|.$$

| N \ p | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| 30 | 0.754 | 0.246 | 0.14 | 0.10 | 0.078 | 0.063 | 0.053 | 0.047 | 0.041 | 0.036 |
| 40 | 0.446 | 0.162 | 0.101 | 0.072 | 0.056 | 0.046 | 0.039 | 0.034 | 0.029 | 0.027 |
| 50 | 0.310 | 0.124 | 0.077 | 0.055 | 0.044 | 0.036 | 0.030 | 0.026 | 0.023 | 0.021 |
| 60 | 0.238 | 0.100 | 0.062 | 0.046 | 0.036 | 0.029 | 0.025 | 0.022 | 0.019 | 0.017 |
| 70 | 0.193 | 0.083 | 0.053 | 0.039 | 0.030 | 0.025 | 0.021 | 0.018 | 0.016 | 0.014 |
| 80 | 0.163 | 0.071 | 0.045 | 0.033 | 0.026 | 0.023 | 0.019 | 0.016 | 0.014 | 0.013 |

TABLE 1
Example in Section 5.1: the index $J$ as a function of the number of nodes $N$ and of the connectivity degree $p$.

To quantify the error filtering property, we have randomly generated networks with a varying connectivity degree. Since the maximum number of internal connections (non-oriented arcs) for a network with $N$ nodes is $m_{max} = \frac{N(N-1)}{2}$, the number of arcs is chosen equal to $m = \frac{N(N-1)}{2}\frac{p}{100}$, where $0 < p \leq 100$ expresses the connectivity degree. To ensure connection, at each random experiment we have initially placed $N$ arcs corresponding to a chain, including an external connection, while the remaining arcs have been added by randomly selecting the departure and arrival node. When, by chance, the two coincide, this corresponds to an external connection. We have then added the noise $\delta$ so that $y = Cx - \delta$, with components $\delta_k$ uniformly generated in $[-1, 1]$, and computed the index $J = \frac{\text{var}(e)}{\text{var}(\delta)}$ at steady state; $n_s = 1000$ samples have been considered. The results for $\gamma = 1$ (which is suitable, in view of Proposition 6) are reported in Table 1. As expected, noise rejection increases (hence, $J$ decreases) with connectivity. Less expectedly, noise rejection increases with $N$, the number of nodes.

### 5.2 Local altitude detection

Consider 16 agents on a surface, communicating according to the connection topology shown in Fig. 3. Each agent exchanges information with its neighbors to determine its own altitude. Two communicating agents, having altitudes $q_i$ and $q_j$, exchange information about their estimated altitudes $z_i$ and $z_j$, and measure the difference $y_{ij} = q_i - q_j$. Being the agents still, the dynamic equations are simply $\dot{q}_i = 0$ for all $i$ and, as in the previous example, our estimation scheme provides the least-square solution.

The first agent only communicates with an anchor (external environment) and could thus, in principle, determine its own (absolute) altitude. However, *the agent is not aware of receiving an absolute altitude from the anchor* and processes the difference between its own altitude and the external reference, assumed as 0, ignoring that it corresponds exactly to its own altitude. Hence, the local observer of agent 1 will have a transient as all of the others. Matrix $C$ has 25 columns, corresponding to all of the arcs in Fig. 3. Fig. 4 shows the evolution of the decentralized altitude estimation, with $\gamma = 5$ and a time horizon of $T = 2$ seconds.

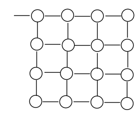

Fig. 3. The altitude setup problem.

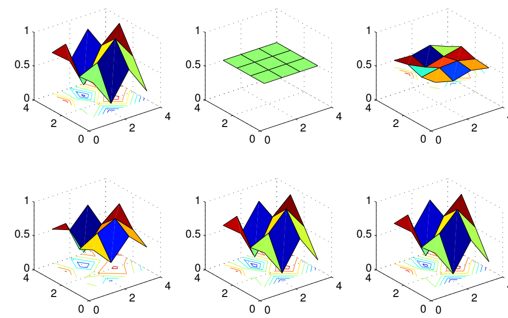

Fig. 4. The true altitude (first frame) and the altitude detection evolution at times 0, 0.01, 0.07, 0.48, 2 seconds (the snapshots are not equidistant to evidence the initial part of the transient).

The first frame shows the actual altitude of the agents, randomly generated between 0 and 1, while frames 2 to 6 represent the estimate evolution: the snapshots are taken at non-equidistant times, to evidence the transient behavior. The initial value of the observer state is chosen with all altitudes equal to $1/2$.

### 5.3 A network of moving agents

Consider the planar motion of agents representing vehicles, or crafts, confined in a square: whenever an agent reaches the boundary of the square, it bounces back. The bounces are elastic (energy-preserving); hence, whenever a bounce occurs, the component of the agent speed that is orthogonal to the hit surface instantaneously changes its sign. This represents a discontinuity, equivalent to introducing disturbance impulses $u_i = \delta_i(t - t_k)$ in the system. Due to these persistent disturbances, the observer error cannot exactly converge to zero. We assume uniform linear motion (zero acceleration and constant speed), apart from bounces instants, and we do not consider other forces (such as friction).

The agents are associated with the nodes of a network representing the communication topology, and want to reconstruct their absolute positions and speeds under the following rules: (i) if two agents $i$ and $j$ can communicate, they measure the relative position $r_i - r_j$ and communicate to each other their own estimated positions $\hat{r}_i$ and $\hat{r}_j$; (ii) if an agent can communicate with an anchor $r_A$, it measures the relative position $r_i - r_A$ and receives the anchor position $\hat{r}_A = r_A$; (iii) the network topology is unknown and may vary: communication is possible only if two nodes are within a maximum distance $\rho_{max}$.

Therefore, the incidence matrix $G_\sigma$ belongs to a family and the overall output matrix $C_\sigma$ is such that $C_\sigma = [G_\sigma^\top \otimes C_1]$. The estimator (3) becomes

$$\begin{aligned}\dot{z}_i &= A_1 z_i + B_1 u_i + \sum_{j \in \mathcal{O}_i} L_1[(\hat{r}_i - \hat{r}_j) - (r_i - r_j)] \\ &= \underbrace{A_1 z_i + L_1 \hat{r}_i + B_1 u_i}_{\text{internal dynamics}} + \sum_{j \in \mathcal{O}_i} \left[ L_1 \left( \underbrace{(r_j - r_i)}_{\text{measured}} - \underbrace{\hat{r}_j}_{\text{received}} \right) \right].\end{aligned}$$

Since anchors are assumed to provide exact information ($\hat{r}_A = r_A$), a node $i$ that communicates with an anchor receives precisely its position: $(r_A - r_i) - \hat{r}_A = r_i$. However, the node is not aware of this fact and uses the received information exactly as if it were communicating with any other node.

The model for each moving agent is:

$$A_1 = \begin{bmatrix} \mathbf{0}_2 & I_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \mathbf{0}_2 \\ I_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} I_2 & \mathbf{0}_2 \end{bmatrix}.$$

For each single subsystem, we consider the Kalman gain $L_1 = -Q_1 C_1^\top$, where $Q_1 = P_1^{-1}$ is the solution of the Riccati equation

$$A_1 Q_1 + Q_1 A_1^\top - Q_1 C_1^\top C_1 Q_1 + M = 0$$

for a suitable $M > 0$. Our approach to network-decentralized estimation provides the overall filter:

$$\dot{z}(t) = (A - \gamma Q C^\top C) z + Bu + \gamma Q C^\top y.$$

In our simulations for a set of 8 agents, with $\rho_{max} = 1.2$, we have solved for each subsystem the Riccati equation with $M = I_4$, obtaining

$$Q_1 = \begin{bmatrix} 1.41 & 0 & 1 & 0 \\ 0 & 1.41 & 0 & 1 \\ 1 & 0 & 1.41 & 0 \\ 0 & 1 & 0 & 1.41 \end{bmatrix}.$$

We have considered three cases, with 1, 2 and 3 anchors; the agents connected to the anchors are associated with the first components of the state vector.

The time evolution of the spacial coordinates of the 8 agents in the case of 2 anchors, with $\gamma = 10$, is reported in Fig. 5, left; the time evolution of the corresponding estimation error, in norm, is in Fig. 5, right.

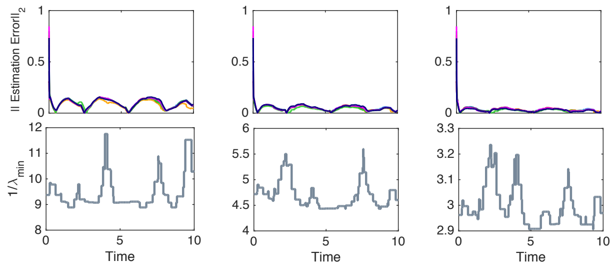

Fig. 6. The time evolution of the error norms (top) and of $1/\lambda_{min}(t)$ (bottom) in the case of 1 (left column), 2 (middle column) and 3 (right column) anchors.

The cases of 1, 2 and 3 anchors, with $\gamma = 15$, are compared in Fig. 6 by plotting the time evolution of the estimation error norm (top row) and of $1/\lambda_{min}(t)$ (bottom row), where $\lambda_{min}(t)$ is the smallest eigenvalue of the generalized Laplacian matrix corresponding to the interconnection configuration at time $t$. As expected, the smallest eigenvalue has greater values when more anchors are present. Note that the choice $\gamma = 15$ does not guarantee that $\gamma > 1/\lambda_{min}$ at any time: stability (hence, a negligible estimation error) is ensured *no matter how the interconnection topology changes during the system evolution* if $\gamma > \sigma_{max}(\Phi_8) \approx 29.37$.

## 6 Conclusions

To design robust network-decentralized observers, such that local agents reconstruct their state by exchanging information with their neighbors only, the smallest eigenvalue $\lambda_{min}$ of the generalized Laplacian matrix of the interconnection graph plays a fundamental role. We have characterized the worst case value of $\lambda_{min}$ and provided a non-conservative lower bound for the observer gain that ensures robust stability even with unknown or switching topologies. We have also shown that, when the system and the measurements are affected by noise, the maximum noise level that can be tolerated depends again on $\lambda_{min}$.

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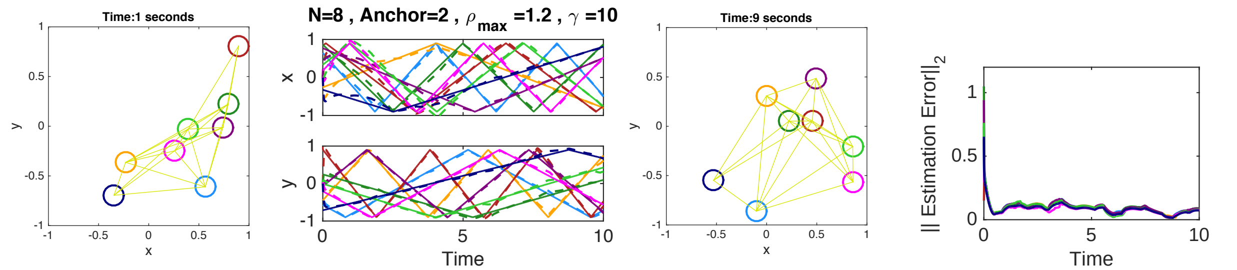


Fig. 5. System with $8$ agents, $2$ anchors and $\gamma = 10$: time evolution of the spacial coordinates $x$ and $y$ (left) and of the norm of the estimation error (right).


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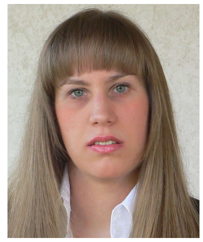

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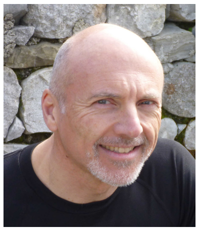

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