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Plant tuning: a robust Lyapunov approach

Franco Blanchini [a], Gianfranco Fenu [b], Giulia Giordano [a], Felice Andrea Pellegrino [b]

***Abstract*— We consider the problem of tuning the output of a static plant whose model is unknown, under the only information that the input–output function is monotonic in each component or, more in general, that its Jacobian belongs to a known polytope of matrices. As a main result, we show that, if the polytope is robustly non–singular (or has full rank, in the non–square case), then a suitable tuning scheme drives the output to a desired point. The proof is based on the application of a well known theorem concerning the existence of a saddle point for a min–max zero–sum game. Some application examples are suggested.**

I. Introduction

In some cases, tuning a plant with several inputs and outputs can be frustrating: when attempting to reach the desired value for some output, the interaction among variables can drive other outputs out of tune.

Consider, for instance, a large electrical network in which the control variables are the voltage generators, and the target is to assure that certain nodes have the appropriate voltage level. If the number of generators (degrees of freedom) is not smaller than the number of the target nodes, then the desired voltage level can be obtained by assigning the proper voltage at the generators. If the network parameters (typically the impedances) are known, the problem has a straightforward solution. Yet this is not always the case, because the network parameters depend on the load, which can vary.

The same applies to many other interconnected plants, such as power generation plants, heat transmission and generation, electronic circuits, and flow networks. For these systems, stability is not the main issue, while steady–state tuning can be a more important task.

In this paper we consider the problem of tuning a *static* plant, described by a system of nonlinear equations. Our main goal is to solve a *Robust Tuning Problem (RTP)*: we want to drive the output to a desired level, when the plant equations are not known.

We show that, if the equations satisfy proper assumptions (precisely, if in some region the Jacobian matrix of the function satisfies proper bounds, which correspond to the inclusion in a polytope of matrices, and if some robust non–singularity conditions are satisfied), then the RTP is solvable. Our main result can be stated as follows.

Assume that the Jacobian of the transformation is included in a polytope of matrices. If all the elements of the polytope are robustly right invertible (in the square case, non–singular), then the Robust Tuning Problem is solvable.

The result requires, as a technical support, the exploitation of the well known min–max theorem in game theory [1]. This property has already been used in the context of robust control via Lyapunov methods [2], [3], [4] and of robust control of flow networks [5]. Here we solve a different problem, since we consider a static plant (in practice, a stable plant with *fast dynamics*) that does not have state variables.

The contributions of the paper are the following.

- We consider a "control scheme" based on an auxiliary control variable chosen as the derivative of the original control (hence, the "state" of the system will be the control variable itself).
- A Lyapunov–like positive–definite function of the output variable is considered, assuming $y = 0$ as the target. It is shown that, by means of a suitable robust control strategy, this function decreases to 0.
- The control, based on a min–max principle, requires the solution of a convex optimization problem on–line.
- The maximum tuning speed can be assigned by constraining the norm of the auxiliary control signal. Under suitable choices of such a norm, the convex optimization is a quadratic problem (Euclidean norm) or a Linear Programming problem (∞–norm).

Some examples are provided to illustrate the technique.

II. Motivating example and problem formulation

Consider the ventilation plant represented in Fig. 1, where a fan forces fresh air in the environment, the air is heated by means of a resistor, and both the current I and the fan speed ω are assumed to be controlled variables. The goal is keeping humidity and temperature at the desired set–point; without restriction, we assume that the desired temperature level and relative humidity level are both 0. We also assume that in the environment there are both thermal dispersion

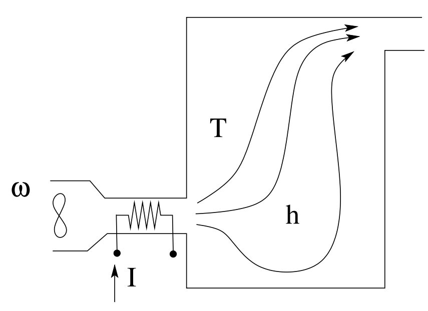

Fig. 1: The heat–humidity regulation problem.

[a] Dipartimento di Matematica e Informatica, Università degli Studi di Udine, 33100 Udine, Italy. blanchini@uniud.it, giulia.giordano@uniud.it
[b] Dipartimento di Ingegneria e Architettura, Università degli Studi di Trieste, 34127 Trieste, Italy. fapellegrino@units.it, fenu@units.it

and a humidity source (for instance due to vegetation). We neglect any delay, namely, we assume that the process is static. Although unrealistic in most circumstances, this is reasonable when the environment is not too large. Then our plant is represented by the equations

$$\begin{aligned} T &= \phi(I,\omega), \\ h &= \psi(I,\omega), \end{aligned}$$

where the functions ϕ and ψ are assumed to be monotonic in both arguments, according to the following trends:

	I	ω
T	+	-
h	-	-

This means we are assuming that:

- the temperature T is a decreasing function of the fan speed ω and an increasing function of the current I;
- the relative humidity h is a decreasing function of both the fan speed ω and the current I.

The precise value of these functions is influenced by unknown factors, such as the external temperature and humidity, and the humidity generation in the environment; only the trend is assumed to be known.

We may also have 0 elements in the Jacobian. For instance, while the relative humidity decreases with the current I, the absolute humidity h_a essentially depends on the external humidity and has practically no dependence on I. Hence, if we considered the absolute humidity h_a instead of h, the trend table would have a 0 in the position $(2,1)$.

Given the Jacobian matrix of the function, we assume that, in a certain domain, the partial derivatives are bounded in absolute value as $\epsilon \leq |(\partial \cdot / \partial \cdot)| \leq \mu$. Then

$$\begin{bmatrix} \frac{\partial T}{\partial I} & \frac{\partial T}{\partial \omega} \\ \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \omega} \end{bmatrix} \in \begin{bmatrix} +m_1 & -m_2 \\ -m_3 & -m_4 \end{bmatrix},$$

where $m_1, m_2, m_3, m_4 \in [\epsilon, \mu]$. We consider a feedback loop as in Fig. 2, where the control has information on the trends and the bounds ϵ and μ only.

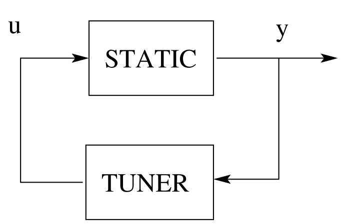

Fig. 2: The automatic tuning.

More in general, we may assume that the Jacobian matrix of the function is included in a polytope of matrices; having upper and lower bounds separately available for each entry is a special case.

A. Problem formulation

We call a *polytope of matrices* a set of the form

$$\mathcal{M} = \{M = \sum_{k=1}^{r} M_k \alpha_k \ : \ \sum_{k=1}^{r} \alpha_k = 1, \ \alpha_k \geq 0\} \quad (1)$$

and we denote by

$$\mathcal{S}_\rho = \{y : y^\top y \leq \rho^2\}$$

the sphere of radius ρ with respect to the Euclidean norm. The problem we wish to consider is the following.

Problem 1: Given the static plant

$$y = g(u), \quad (2)$$

where $g : \mathbb{R}^q \to \mathbb{R}^p$, $p \leq q$, assume that, for some $\bar{u}$, $g(\bar{u}) = 0$ and that the following inclusion holds:

$$G_u \doteq \left[\frac{\partial g}{\partial u}\right] \in \mathcal{M}. \quad (3)$$

Find a dynamic algorithm such that

$$y(t) \to 0 \quad (4)$$

and

$$u(t) \to \bar{u} \quad (5)$$

as $t \to \infty$, where $\bar{u}$ solves the equation

$$0 = g(u). \quad (6)$$

◇

As a first attempt to solve the problem, we may consider the system $\dot{x}(t) = g(u)$ and then drive x to zero. This would be possible in principle by adopting the techniques presented hereafter. However, the control u may be discontinuous in time. In the next section, we will propose a dynamic solution ensuring a continuous evolution of the inputs (beneficial, for instance, to avoid exciting high–frequency unmodeled dynamics of the plant).

The considered problem is related to methods that have been previously proposed for parameter tuning [12], aimed at optimizing performances and/or identifying the system parameters. Also, a different approach to solve the problem could rely on multi–dimensional extremum seeking techniques [13], [14], [15]. Indeed, our goal is reached when $\|g(u)\|^2 = 0$; thus, in principle, the problem could be formulated as extremum seeking. However, the substantial difference of our approach is that we are exploiting the structure of g and its Jacobian, information that would be lost in the extremum–seeking framework.

III. Solution of the problem

A. The case $p = q$

We consider here Problem 1 under the assumption that there are as many outputs as inputs, hence $p = q$.

Since we are assuming that condition (3) is the only available information for control purposes, we need to design a *robust* scheme. We just need the further assumption.

Assumption 1: **Robust non–singularity.** Any matrix in the polytope $\mathcal{M}$ is non–singular.

For instance, the matrix associated with the heat–humidity problem in Section II is robustly non–singular. We will see later how the condition in Assumption 1 can be checked, under suitable assumptions on the matrix structure.

Theorem 1: Under Assumption 1, Problem 1 can be solved with a control scheme of the form

$$\dot{u}(t) = v(t), \quad (7)$$

$$v(t) = \Phi(y(t)), \quad (8)$$

with added input variable $v(t)$. □

The constructive proof of the theorem requires several preliminary steps, which are described next.

B. Proof of Theorem 1

First, for both technical and practical reasons, we deliberately bound the control as

$$\|v(t)\| \leq \xi, \quad (9)$$

where $\xi > 0$ and $\|\cdot\|$ is any norm.

As a second step, we momentarily *pretend* that the Jacobian G_u in u is available to the controller (which is not true). Namely, instead of the control action (8), we assume to have $v(t) = \Phi(y(t), G_u)$.

We consider the Lyapunov–like positive definite function

$$V(y) = \frac{1}{2} y^\top y,$$

whose Lyapunov derivative is

$$\dot{V} = y^\top \dot{y} = y^\top G_u \dot{u} = y^\top G_u v. \quad (10)$$

Then, since G_u is invertible, we take the "fake" control

$$v = -\gamma(y) G_u^{-1} y, \quad (11)$$

where $\gamma(y) > 0$ is a suitable continuous scalar function, to get

$$\dot{V} = -\gamma(y) y^\top y < 0, \quad \text{for} \quad y \neq 0. \quad (12)$$

We can choose function γ so as to ensure

$$\|v\| = \|\gamma(y) G_u^{-1} y\| \leq \xi, \quad (13)$$

therefore achieving the following preliminary result.

Lemma 1: The control (11) satisfies (9) and asymptotically drives $y(t)$ to 0. □

The next step towards the *true* solution requires a game–theoretic interpretation of (10), (12) and (13).

If y and $\gamma(\cdot)$ are given, then the following holds:

Statement 1 For all $G_u \in \mathcal{M}$, there exists v, $\|v\| \leq \xi$, such that $\dot{V} \leq -\gamma(y) y^\top y$.

This is equivalent to writing

$$\max_{G_u \in \mathcal{M}} \min_{\|v\| \leq \xi} y^\top G_u v \leq -\gamma(y) y^\top y.$$

Since $y^\top G_u v$ is bilinear in $G_u \in \mathcal{M}$ and $v \in \mathcal{S}_\xi$, and the sets $\mathcal{M}$ and $\mathcal{S}_\xi$ are compact and convex, a fundamental theorem in game theory [1] states that min and max commute, *i.e.*, the min–max game has a saddle point. Hence also

$$\min_{\|v\| \leq \xi} \max_{G_u \in \mathcal{M}} y^\top G_u v \leq -\gamma(y) y^\top y.$$

Thus, Statement 1 is equivalent to the following.

Statement 2 There exists v, $\|v\| \leq \xi$, such that, for all $G_u \in \mathcal{M}$, $\dot{V} \leq -\gamma(y) y^\top y$.

Consider $M^* \in \mathcal{M}$ and $v^* \in \mathcal{S}_\xi$ such that $(v^*(y), M^*(y))$ is the saddle point of

$$\max_{G_u \in \mathcal{M}} \min_{\|v\| \leq \xi} y^\top G_u v = \min_{\|v\| \leq \xi} \max_{G_u \in \mathcal{M}} y^\top G_u v \quad (14)$$

$(= y^\top M^*(y) v^*(y))$ and define the discontinuous control

$$\Phi(y) \doteq v^*(y). \quad (15)$$

Then we have

$$\dot{V} = y^\top G_u \Phi(y) = y^\top G_u v^*(y) \leq -\gamma(y) y^\top y. \quad (16)$$

for all G_u. This condition assures that, if in (8) we take $\Phi(y)$ as in (15), the control (7)–(8) guarantees that $y(t) \to 0$. Moreover, since u is the integral function of v, which is a continuous function, we have $u(t) \to \bar{u}$, where $g(\bar{u}) = 0$.

As far as convergence is concerned, from (16) we have

$$\frac{\dot{V}}{V} \leq -2\gamma(y),$$

hence

$$\frac{d}{dt} \log V \leq -2\gamma(y).$$

Then, if $\gamma \geq \bar{\gamma}$, we have exponential convergence:

$$V(y(t)) \leq V(y(0)) e^{-2\bar{\gamma} t}.$$

Theorem 1 is therefore proved.

Remark 1: The dynamics of the output y can be described by $\dot{y} = G_u v$. If we assume invertibility of g, so that $u = g^{-1}(y)$, we have that y is represented by a driftless system [6], [7], for which several stabilizability results are available. These results do not apply to our case, since we assume the model completely unknown. Note however that we have some analogies, since also in our case we resort to a discontinuous control, as it must be done for driftless systems [6], [7].

C. The case $p < q$

If the number of inputs is lesser than the number of outputs, Assumption 1 needs to be changed as follows.

Assumption 2: **Robust right invertibility.** Any matrix in the polytope $\mathcal{M}$ is right invertible.

Then, we can modify (11) by simply taking the pseudo inverse instead of the inverse,

$$v = -\gamma(y) G_u^{\perp} y, \quad (17)$$

and, along the same reasoning as in the previous subsection, we reach the same conclusions. It is worth noticing that there are in general multiple solutions to $g(u) = 0$ and the final value u will depend on the initialization.

Remark 2: We have excluded from the formulation of Problem 1 the problematic case in which the number of inputs is greater than the number of outputs, because, if $p > q$, a solution to $g(u) = 0$ does not exist in general. Typically, in this case, it is possible to choose a suitable function $h(y)$ of y and drive $h(y)$ to zero.

IV. Implementation of the Scheme

For implementing the scheme, two steps are required.

- **Off–line** Checking the robust non–singularity (or rank completeness) of the polytope of matrices.
- **On–line** Computing the tuning law.

A. Checking robust non–singularity (or rank completeness)

In the case $p = q$, before implementing the tuning scheme we need to make sure that the given polytope of matrices $\mathcal{M}$ is robustly non–singular.

Checking non–singularity is a hard problem in general [8], especially for high dimensional systems. For reasonable instances, however, this task can be computationally tractable and remarkable solutions are available, as shown next.

Proposition 1: [9], [10] **Rank one generating matrices.** If

$$M = \sum_{i=1}^{r} d_i M_i, \quad d_i^- \leq d_i \leq d_i^+,$$

where M_i are rank one matrices, then robust non–singularity is equivalent to robust non–singularity of all the vertices:

$$\det \left[\sum_{i(\pm)} d_i^{\pm} M_i \right] \neq 0$$

where the sum means, with an abuse of notation, that we take the coefficients d_i on the extrema of their intervals, obtaining 2^r possible combinations.[1] □

Proposition 2: **Interval matrices** Assume that we are dealing with an interval matrix M, having entries

$$M_{ij}^- \leq M_{ij} \leq M_{ij}^+.$$

Then robust non–singularity is equivalent to robust non–singularity of all the vertices. □

Conversely, for $p \neq q$, in general we must check that all the matrices of the family have full rank. One obvious possibility is checking if there is at least one full size square non–singular sub–matrix.

For particular systems, non–singularity can be inferred from the structure. Consider for instance the flow system

$$g(u) = Bh(u) + b_0, \tag{18}$$

where $B \in \mathbb{R}^{p \times q}$, $b_0 \in \mathbb{R}^p$ and

$$h(u) = [h_1(u_1)\ h_2(u_2) \ldots h_q(u_q)]^\top$$

is a componentwise strictly increasing function such that

$$0 < \eta^- \leq h_i'(u_i) \leq \eta^+.$$

Denoting by $\eta = h'(u)$, the Jacobian matrix is

$$M_u = B\eta. \tag{19}$$

Then we have the following.

Proposition 3: M_u as in (19) is non–singular (if $p = q$; otherwise, full rank) if and only if B is non–singular (respectively, full rank). □

Proof: If $p = q$, $M_u = B\eta$ has a non–trivial kernel $\ker(B\eta) = \{z : [B\eta]z = 0\}$ iff B has a non–trivial kernel. The proof for $p \neq q$ is similar, but tedious, and is omitted for space reasons. ■

[1]For instance, if $r = 2$: (d_1^-, d_2^-), (d_1^+, d_2^-), (d_1^-, d_2^+), (d_1^+, d_2^+).

B. Computing the tuning law Φ

To compute the control law (15), we need to solve the min–max problem (14) on–line. The saddle point value is computed as follows. Remind that the matrices in the polytope (1) are $M = \sum_{k=1}^r M_k \alpha_k$, with $\alpha \in \mathcal{A} = \{\hat{\alpha} : \sum_{k=1}^r \hat{\alpha}_k = 1, \ \hat{\alpha}_k \geq 0\}$. Then we need to compute

$$J = \max_{\alpha \in \mathcal{A}} \left[\min_{\|v\| \leq \xi} \sum_{k=1}^{r} \alpha_k y^\top M_k v \right]. \tag{20}$$

Let y be given and, to simplify the notation, define

$$z_i^\top(y) \doteq y^\top M_i.$$

(i) If the control is bounded by the Euclidean norm, then, for given α, the minimizer in (20) is

$$v(\alpha) = -\xi \frac{\sum_{k=1}^r \alpha_k z_k^\top(y)}{\| \sum_{k=1}^r \alpha_k z_k^\top(y)\|}. \tag{21}$$

Plugging the value of $v(\alpha)$ in (20) leads to

$$J = -\xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\|.$$

The optimal strategy for the maximizer α is to take

$$\alpha^*(y) = \max_{\alpha \in \mathcal{A}} -\xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\| = -\min_{\alpha \in \mathcal{A}} \xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\|.$$

This is a convex optimization problem (determining the smallest point in a polytope, see [11]). Once $\alpha^*(y)$ is found, $v^* = v(\alpha^*)$ can be computed as in (21). Then the control is

$$\Phi(y) = v(\alpha^*(y)).$$

(ii) If the control is bounded by the infinity norm, then, for a given α, the minimizer in (20) is

$$v(\alpha) = -\xi \text{sign} \left[\sum_{k=1}^{r} \alpha_k z_k^\top(y) \right], \tag{22}$$

where sign[·] is the componentwise sign vector. Plugging this new value of $v(\alpha)$ in (20) leads to

$$J = -\xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\|_1.$$

The optimal strategy for α is to take

$$\begin{aligned} \alpha^* &= \max_{\alpha \in \mathcal{A}} -\xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\|_1 \\ &= -\min_{\alpha \in \mathcal{A}} \xi \left\| \sum_{k=1}^{r} \alpha_k z_k^\top(y) \right\|_1, \end{aligned} \tag{23}$$

which reduces to a linear programming problem. Given the non–negative unknown vectors $w^+, w^- \in \mathbb{R}^p$, (23) can be solved as

$$\min \left[\sum_i w_i^+ + \sum_i w_i^-\right] \tag{24}$$

$$\text{s.t.} \quad \sum_{k=1}^{r} \alpha_k z_k^\top(y) = w^+ - w^-, \tag{25}$$

$$w_i^+ \geq 0, \quad w_i^- \geq 0. \tag{26}$$

Once α^* is found, the optimal value $v^* = v(\alpha^*)$ can be computed as in (22). Again, $\Phi(y) = v(\alpha^*(y))$.

V. EXAMPLES

A. The heat–humidity regulation problem

Reconsider the example in Section II. The Jacobian of the transformation can be included in the interval matrix

$$\begin{bmatrix} +m_1 & -m_2 \\ -m_3 & -m_4 \end{bmatrix},$$

where $m_1, m_2, m_3, m_4 \in [\epsilon, \mu]$ and ϵ, μ are known. We assume the control is bounded by the Euclidean norm. For any $y = [y_1\ y_2]^\top$, we need to solve $\min \|y^\top M\|$:

$$\min_{\epsilon \leq m_i \leq \mu} \|[m_1 y_1 - m_3 y_2 \quad - m_2 y_1 - m_4 y_2]\|.$$

In this case, the problem can be split into two separate problems:

$$\min_{\epsilon \leq m_1, m_3 \leq \mu} (m_1 y_1 - m_3 y_2)^2, \tag{27}$$

$$\min_{\epsilon \leq m_2, m_4 \leq \mu} (m_2 y_1 + m_4 y_2)^2. \tag{28}$$

These are elementary problems, and the square root of the sum of the optimal values is the optimum.

The minimizer of (27) (resp. of (28)) is given by the point, in the square $\epsilon \leq m_1, m_3 \leq \mu$ (resp. $\epsilon \leq m_2, m_4 \leq \mu$), which is closest to the line $L_A : m_1 y_1 - m_3 y_2 = 0$ (resp. $L_B : m_2 y_1 + m_4 y_2 = 0$). The two lines are orthogonal if represented in a Cartesian coordinate system where m_1 and m_4 are reported in abscissa and m_2 and m_3 in ordinate, as in Fig. 3. Hence, they cannot give simultaneously 0 as optimal cost.

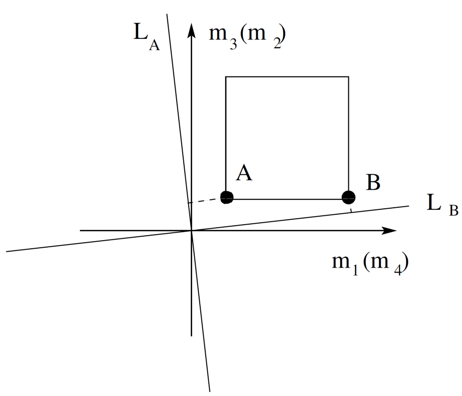

Fig. 3: Optimal points in the m_1–m_3 (m_2–m_4) space.

Then the control v to be used is

$$v = -\xi \frac{[m_1^* y_1 - m_3^* y_2 \quad - m_2^* y_1 - m_4^* y_2]^\top}{\|[m_1^* y_1 - m_3^* y_2 \quad - m_2^* y_1 - m_4^* y_2]^\top\|}.$$

In our simulation we take the functions (unknown to the controller)

$$\Delta T = (0.1 I^2 - 0.5\omega) - 20,$$
$$\Delta h = (-2\ 10^{-4} I^2 - 2.5\ 10^{-2}\omega + 0.9) - 0.5,$$

where $\Delta T = 0\,°\mathrm{C}$ (corresponding to $T = 20\,°\mathrm{C}$) is the desired temperature, and $\Delta h = 0$ (corresponding to $h = 0.50$) is the desired relative humidity. Note that for $\omega = 0$ and $I = 0$ we would obtain $\Delta T = -20\,°\mathrm{C}$, corresponding to a temperature $T = 0\,°\mathrm{C}$, and $\Delta h = 0.40$, corresponding to relative humidity $h = 0.90$.

By solving a linear equation in the unknowns I^2 and ω, we obtain that the equilibrium inputs that achieve the desired values are $\bar{I} = \bar{u}_1 = 16.41\,\mathrm{A}$ and $\bar{\omega} = \bar{u}_2 = 13.85\,\mathrm{rad/s}$. These values are unknown to the controller (as the whole model is); yet, by choosing the rough bounds $\epsilon = 10^{-3}$ and $\mu = 200$, and applying the control strategy described above with $\xi = 5$, we obtain the temperature and humidity transients represented in Fig. 4 (the initial conditions being $T = 10\,°\mathrm{C}$ and $h = 0.8$). The values of the control converge to the equilibrium pair, as shown in Fig. 5. A dead zone of $|\Delta T| \leq 0.1\,°\mathrm{C}$ and $\Delta h \leq 0.01$ has been employed; namely, the control is turned off when the output is within the dead-zone.

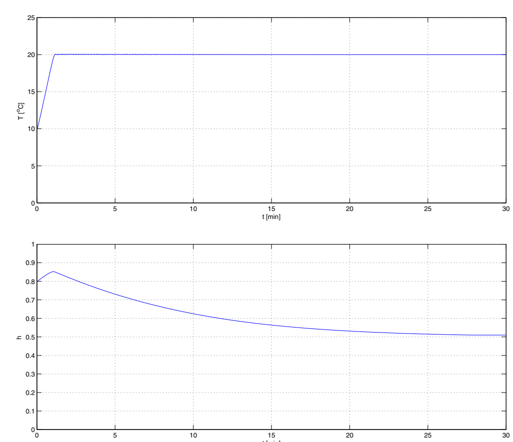

Fig. 4: Transient of the temperature T [°C] (top) and of the relative humidity h (bottom) for the heat–humidity problem.

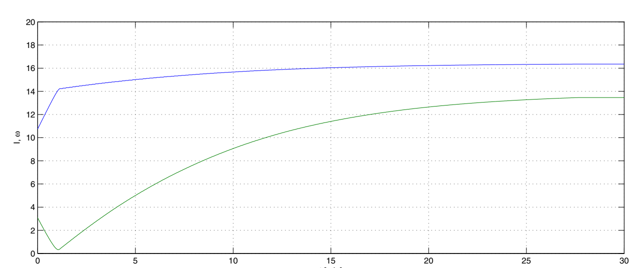

Fig. 5: Input variables during a transient of the heat-humidity problem: $u_1 = I$ (current) is represented in blue, $u_2 = \omega$ (fan speed) in green.

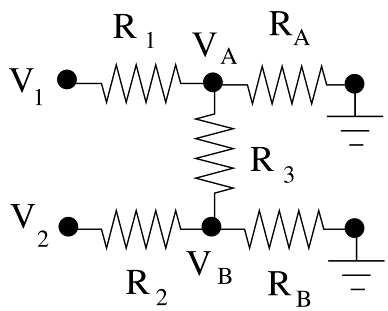

Fig. 6: A simple circuit.

B. Electric voltage tuning

Consider the electric circuit in Fig. 6, where the voltages V_1 and V_2 have to be selected in order to keep V_A and V_B to the desired levels:

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \frac{R_1}{R_A} + \frac{R_1}{R_3} & -\frac{R_1}{R_3} \\ -\frac{R_2}{R_3} & 1 + \frac{R_2}{R_B} + \frac{R_2}{R_3} \end{bmatrix}}_{\doteq P}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

By computing $M = P^{-1}$ and posing $y_1 = V_A - \bar{V}_A$ and $y_2 = V_B - \bar{V}_B$, we get the representation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a+c+d & b \\ c & e+b+d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \bar{V}_A \\ \bar{V}_B \end{bmatrix},$$

where a, b, c, d, e are all strictly positive, being of the form

$$\frac{R_i R_j R_k}{(\det P)(R_A R_B R_3)}, \quad i, j, k \in \{1, 2, 3, A, B\}.$$

Assuming $0 < \epsilon \leq a, b, c, d, e \leq \mu$, the Jacobian

$$M = \begin{bmatrix} a+c+d & b \\ c & e+b+d \end{bmatrix}$$

is robustly non–singular. To obtain an auxiliary control that is bounded in the infinity norm, *i.e.*, $||v||_\infty \leq \xi$, we apply (22) taking α as in (23). In detail, given y, we need to find a^*, b^*, c^*, d^*, e^* that minimize

$$||y^\top M||_1 = |y_1(a+b+c) + y_2 c| + |y_1 b + y_2(e+b+d)|$$

in the domain

$$\epsilon \leq a, b, c, d, e \leq \mu.$$

In general the minimizer M^* can be found by solving the linear program (24)–(26). Then we apply

$$v = -\xi \text{sign}[y^\top M^*].$$

We take $R_1 = R_A = 10k\Omega$, $R_2 = 15k\Omega$, $R_3 = 9k\Omega$, $R_B = 7.5k\Omega$, obtaining a matrix (unknown to the controller)

$$M = \begin{bmatrix} 0.368 & 0.0877 \\ 0.132 & 0.246 \end{bmatrix}.$$

If the desired output values are $\bar{V}_A = 1\ V$, $\bar{V}_B = 2\ V$, by solving the linear system $\bar{u} = M\bar{y}$, with $\bar{y} = [\bar{V}_A \quad \bar{V}_B]^\top$, we find the control equilibrium values $\bar{u}_1 = 0.89\ V$ and $\bar{u}_2 = 7.67\ V$. For null initial conditions, the proposed control scheme leads to the desired voltages, without knowledge of M. By taking $\epsilon = 10^{-2}$, $\mu = 5$, and $\xi = 5$, we obtain the transient reported in Figures 7 and 8.

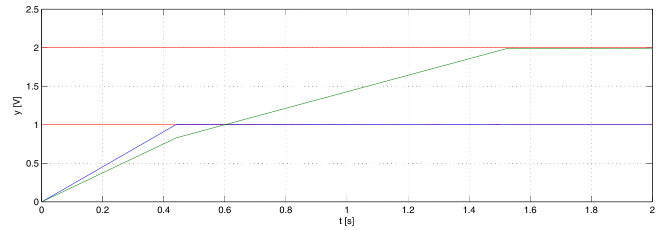

Fig. 7: Output variables during a transient of voltage tuning: y_1 [V] (blue) and y_2 [V] (green). The desired levels are represented in red.

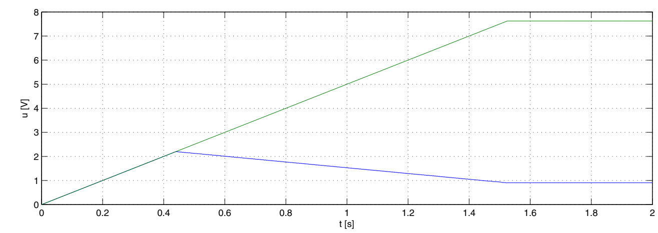

Fig. 8: Input variables during a transient of voltage tuning: u_1 [V] (blue), u_2 [V] (green).

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