

Optimal Control of Epidemic Models with Discrete Time Delays

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“Sakawa-Shindo algorithm for optimal control of time-delay systems, with applications to epidemiology”

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Incorporating discrete time delays into epidemic models gives rise to significant analytical and computational challenges.

Although Pontryagin’s minimum principle admits an extension to delay differential equation (DDE) systems, the resulting optimality conditions lead to boundary value problems involving both forward- and backward-running delays, which are seldom tractable in closed form. This has motivated the development of a variety of numerical approaches to approximate solutions of optimal control problems (OCPs) with time delays, broadly falling into two categories: indirect methods, which discretize the optimality conditions, and direct methods, which reformulate the continuous problem as a finite-dimensional nonlinear program.

Indirect methods proceed from the necessary conditions derived within the Hamiltonian framework for delay systems. A rigorous treatment of OCPs in which delays appear in both state and control variables, under mixed control–state constraints, was provided in [1], which also discussed discretization strategies to convert the delayed optimal control problem into a large-scale nonlinear programming problem. Drawing on variational principles, [2] reformulates the problem as a delay-free OCP governed by a system of partial differential equations and proposes a spectral collocation scheme for functional differential equation control systems, expanding the state and control trajectories in Chebyshev polynomial bases and enforcing delay constraints at collocation nodes. A computationally efficient alternative is offered by finite difference discretizations of the state and costate equations; in this vein, [3] develops a θ method for state-delayed OCPs that interpolates between explicit and implicit time integration schemes.

Direct methods, by contrast, bypass the explicit derivation of optimality conditions by optimizing simultaneously over discretized state and control trajectories. For instance, in [4], the authors introduce a simultaneous direct method tailored to DDE-constrained OCPs, based on a linearization of the delayed state within each time subinterval.

These techniques have found concrete application in the optimization of vaccination and intervention strategies within epidemiological frameworks. [5] considered a time-delayed SIR model with a vaccination term and solved a numerical optimal control problem via the forward-backward sweep (FBS) algorithm [6], determining rollout strategies that minimize cumulative infections subject to supply constraints. In a related contribution, [7] formulated a delayed vaccination control problem for an epidemic model using a feedback linearization technique, enabling the design of a switching vaccination strategy and validating it against COVID-19 incidence data. The present work aims to contribute to this body of literature by studying an extension of the Sakawa–Shindo algorithm [8] — originally developed for delay-free OCPs — to the setting of optimal control problems governed by delay differential equations with discrete time delays.

1 Optimal Control Problem with Discrete Delays

We consider a general optimal control problem with discrete time delays of the form:

$$\begin{aligned}
 \min_{u \in \mathcal{U}} \quad & J(x, u) = \gamma(x(T)) + \int_{t_0}^T \ell(t, x(t-h_0), x(t-h_1), \dots, x(t-h_k), u(t)) dt \\
 \text{subject to} \quad & \dot{x}(t) = f(t, x(t-h_0), x(t-h_1), \dots, x(t-h_k), u(t)), \quad t \in [t_0, T], \\
 & x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-h, 0], \\
 & u(t) \in \mathcal{U} \quad \text{a.e. } t \in [t_0, T],
 \end{aligned} \tag{1}$$

where

- $x \in \mathbb{R}^n$ is the state vector (which, for the considered epidemiological models, denotes the fraction of population in each of the compartments);
- $u \in \mathbb{R}^m$ is the control vector (which, for the considered epidemiological models, denotes the intensity of intervention strategies);
- $\mathcal{U}$ is the set of admissible controls;
- $h_0 = 0$ and $h_i > 0$, $i \in \{1, \dots, k\}$, are constant time delays;
- $h = \max_{i \in \{1, \dots, k\}} h_i$;
- $\gamma: \mathbb{R}^n \rightarrow \mathbb{R}$ is the terminal cost;
- $\ell: [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is the running cost.

We adopt the notation $x_{h_i} := x(t-h_i)$ and $X_h(t) := (x(t-h_0), x(t-h_1), \dots, x(t-h_k)) \in \mathbb{R}^{(k+1)n}$, so that the optimal control problem (1) can be rewritten as

$$\begin{aligned}
 \min_{u \in \mathcal{U}} \quad & \gamma(x(T)) + \int_{t_0}^T \ell(t, X_h(t), u(t)) dt \\
 \text{subject to} \quad & \dot{x}(t) = f(t, X_h(t), u(t)), \quad t \in [t_0, T], \\
 & x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-h, 0], \\
 & u(t) \in \mathcal{U} \quad \text{a.e. } t \in [t_0, T],
 \end{aligned}$$

2 Pontryagin's Minimum Principle with Delays

We state Pontryagin's Minimum Principle for systems with discrete time delays.

Theorem 2.1 (PMP for Delay Systems). *Let (x^*, u^*) be an optimal pair for the optimal control problem (1). Then, there exists a costate function $\lambda: [t_0, T] \rightarrow \mathbb{R}^n$ such that, defining the Hamiltonian as*

$$\mathcal{H}(t, X_h, u, \lambda) = \ell(t, X_h, u) + \lambda^\top f(t, X_h, u),$$

and recalling that $X_h^(t) = (x^*(t), x^*(t - h_1), \dots, x^*(t - h_k))$, we have the following:*

1. **State equation** (forward in time):

$$\dot{x}^*(t) = f(t, X_h^*(t), u^*(t)),$$

with initial condition

$$x^*(t_0 + \theta) = \phi(\theta),$$

2. **Costate equation** (backward in time):

$$\begin{aligned} -\dot{\lambda}(t) &= \sum_{i=0}^k \mathbf{1}_{[t_0, T-h_i]}(t) \cdot \frac{\partial \mathcal{H}}{\partial x_{h_i}}(t + h_i, X_h^*(t - h_k + h_i), u^*(t + h_i)) \\ &= \frac{\partial \ell}{\partial x}(t, x^*(t), u^*(t)) + \sum_{i=0}^k \mathbf{1}_{[t_0, T-h_i]}(t) \cdot \frac{\partial f}{\partial x_{h_i}}(t + h_i, X_h^*(t - h_k + h_i), u^*(t + h_i))^\top \lambda(t + h_i) \end{aligned} \quad (2)$$

with terminal condition:

$$\lambda(T) = \frac{\partial \gamma}{\partial x}(x^*(T))$$

3. **Optimality condition** (Hamiltonian minimization):

$$u^*(t) \in \arg \min_{u \in \mathcal{U}} \mathcal{H}(t, X_h^*(t), u, \lambda(t))$$

Remark 2.2. *The advanced time terms in equation (2) account for the influence of the state at time t on the dynamics at time $t + h_i$ through the delay term. The indicator function $\mathbf{1}_{[t_0, T-h_i]}(t)$ ensures that the corresponding term only appears when $t + h_i \leq T$.*

3 Mathematical Models and Optimality Systems

We present a selection of epidemiological models affected by time delays, derive their optimality systems using Pontryagin's Minimum Principle for delay systems, and establish the necessary conditions for optimal control. Recall that we denote the delayed states using the notation $I_h = I(t - h)$. We also denote by P_U the projection operator over the set U .

3.1 SIR Model with Delay

The SIR model considers susceptible (S), infected (I) and recovered (R) individuals as state variables; $\Lambda \geq 0$ is the birth rate, $\mu_i \geq 0$ is the natural death rate for compartment i (where $i \in \{S, I, R\}$), β is the transmission rate (or contagion rate), $\gamma > 0$ is the recovery rate from infection, while $h > 0$ is the delay associated with the incubation period. The control variable $u(t) \in [0, u_{\max}]$ represents the stringency of interventions, such as physical distancing or mask mandates, that reduce the transmission rate β . The corresponding dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta[1 - u(t)]S(t)I(t - h) - \mu_S S(t) \\ \dot{I}(t) = \beta[1 - u(t)]S(t)I(t - h) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \mu_R R(t) \end{cases} \quad (3)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2}I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (3)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, I, R, S_h, I_h, R_h, u, \lambda_S, \lambda_I, \lambda_R)$ is

$$\begin{aligned} \mathcal{H} &= w_I I + w_u u^2 + \lambda_S (\Lambda - \beta(1 - u)SI_h - \mu_S S) + \lambda_I (\beta(1 - u)SI_h - \gamma I - \mu_I I) + \lambda_R (\gamma I - \mu_R R) \\ &= w_I I + \lambda_S (\Lambda - \mu_S S) - \lambda_I (\gamma I + \mu_I I) + \lambda_R (\gamma I - \mu_R R) + w_u u^2 + \beta(1 - u)SI_h(\lambda_I - \lambda_S). \end{aligned}$$

Proposition 3.1 (SIR Optimality System). *Let (S^*, I^*, R^*, u^*) be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta(1 - u^*(t))I^*(t - h)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I)\lambda_I(t) + \gamma\lambda_R(t) \\ &\quad + \mathbf{1}_{[t_0, T-h]}(t) [\beta(1 - u^*(t + h))S^*(t + h)(\lambda_I(t + h) - \lambda_S(t + h))], \\ -\dot{\lambda}_R(t) &= -\mu_R \lambda_R(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{\beta S^*(t)I^*(t - h)(\lambda_I(t) - \lambda_S(t))}{2w_u} \right].$$

3.2 SIRS Model with Delay

The SIRS model extends the SIR model by accounting for waning immunity, due to which recovered individuals become again susceptible at rate $\delta > 0$. The resulting dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta[1 - u(t)]S(t)I(t-h) + \delta R(t) - \mu_S S(t) \\ \dot{I}(t) = \beta[1 - u(t)]S(t)I(t-h) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \delta R(t) - \mu_R R(t) \end{cases} \quad (4)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2}I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (4)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H}(S, I, R, S_h, I_h, R_h, u, \lambda_S, \lambda_I, \lambda_R) = & w_I I + w_u u^2 + \lambda_S (\Lambda - \beta(1 - u)SI_h + \delta R - \mu_S S) \\ & + \lambda_I (\beta(1 - u)SI_h - \gamma I - \mu_I I) + \lambda_R (\gamma I - \delta R - \mu_R R). \end{aligned}$$

Proposition 3.2 (SIRS Optimality System). *Let (S^*, I^*, R^*, u^*) be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta(1 - u^*(t))I^*(t-h)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I)\lambda_I(t) + \gamma\lambda_R(t) \\ &\quad + \mathbf{1}_{[t_0, T-h]}(t) [\beta(1 - u^*(t+h))S^*(t+h)(\lambda_I(t+h) - \lambda_S(t+h))], \\ -\dot{\lambda}_R(t) &= \delta\lambda_S(t) - (\mu_R + \delta)\lambda_R(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{\beta S^*(t)I^*(t-h)(\lambda_I(t) - \lambda_S(t))}{2w_u} \right].$$

3.3 SEIRS Model with Delay

With respect to the SIRS model, the SEIRS model further introduces an exposed compartment $E(t)$ representing individuals who are infected, but not yet infectious. Exposed individuals become infectious at rate $\alpha > 0$. The corresponding dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta[1 - u(t)]S(t)I(t-h) + \delta R(t) - \mu_S S(t) \\ \dot{E}(t) = \beta[1 - u(t)]S(t)I(t-h) - \alpha E(t) - \mu_E E(t) \\ \dot{I}(t) = \alpha E(t) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \delta R(t) - \mu_R R(t) \end{cases} \quad (5)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2}E(T)^2 + \frac{1}{2}I(T)^2 + \int_{t_0}^T (w_E E(t) + w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (5)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, E, I, R, S_h, E_h, I_h, R_h, u, \lambda_S, \lambda_E, \lambda_I, \lambda_R)$ is

$$\begin{aligned} \mathcal{H} = & w_E E + w_I I + w_u u^2 + \lambda_S (\Lambda - \beta(1 - u)SI_h + \delta R - \mu_S S) \\ & + \lambda_E (\beta(1 - u)SI_h - \alpha E - \mu_E E) + \lambda_I (\alpha E - \gamma I - \mu_I I) + \lambda_R (\gamma I - \delta R - \mu_R R). \end{aligned}$$

Proposition 3.3 (SEIRS Optimality System). *Let $(S^*, E^*, I^*, R^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_E, \lambda_I, \lambda_R$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta(1 - u^*(t))I^*(t-h)(\lambda_E(t) - \lambda_S(t)) - \mu_S \lambda_S(t), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I)\lambda_I(t) + \gamma\lambda_R(t) \\ &\quad + \mathbf{1}_{[t_0, T-h]}(t) [\beta(1 - u^*(t+h))S^*(t+h)(\lambda_E(t+h) - \lambda_S(t+h))], \\ -\dot{\lambda}_E(t) &= w_E + \alpha(\lambda_I(t) - \lambda_E(t)) - \mu_E \lambda_E(t) \\ -\dot{\lambda}_R(t) &= \delta\lambda_S(t) - (\mu_R + \delta)\lambda_R(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_E(T) = E^*(T), \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{\beta S^*(t)I^*(t-h)(\lambda_E(t) - \lambda_S(t))}{2w_u} \right].$$

3.4 SIRV Model with Delay

The SIRV model extends the SIR model by adding a vaccinated compartment $V(t)$; the control variable $u(t) \in [0, u_{\max}]$ now represents the vaccination rate. The system equations are

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t-h) - u(t)S(t) - \mu_S S(t) \\ \dot{I}(t) = \beta S(t)I(t-h) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \mu_R R(t) \\ \dot{V}(t) = u(t)S(t) - \mu_V V(t) \end{cases} \quad (6)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2}I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (6)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, I, R, V, S_h, I_h, R_h, V_h, u, \lambda_S, \lambda_I, \lambda_R, \lambda_V)$ is

$$\begin{aligned} \mathcal{H} = & w_I I + w_u u^2 + \lambda_S (\Lambda - \beta S I_h - u S - \mu_S S) + \lambda_I (\beta S I_h - \gamma I - \mu_I I) \\ & + \lambda_R (\gamma I - \mu_R R) + \lambda_V (u S - \mu_V V). \end{aligned}$$

Proposition 3.4 (SIRV Optimality System). *Let $(S^*, I^*, R^*, V^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R, \lambda_V$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta I^*(t-h)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t) - u^*(t)(\lambda_S(t) - \lambda_V(t)), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I)\lambda_I(t) + \gamma \lambda_R(t) + \mathbf{1}_{[t_0, T-h]}(t) [\beta S^*(t+h)(\lambda_I(t+h) - \lambda_S(t+h))], \\ -\dot{\lambda}_R(t) &= -\mu_R \lambda_R(t), \\ -\dot{\lambda}_V(t) &= -\mu_V \lambda_V(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0, \quad \lambda_V(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{S^*(t)(\lambda_S(t) - \lambda_V(t))}{2w_u} \right].$$

3.5 Extended SIRV Model with Delay

This model extends the SIRV model by further incorporating waning immunity, for both recovered and vaccinated, at rates $\sigma_R > 0$ and $\sigma_V > 0$ respectively, and the possibility of infection for both recovered (reinfection, with probability $\theta_R \in [0, 1]$) and vaccinated (imperfect vaccination, with probability $\theta_V \in [0, 1]$). The corresponding dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t-h) + \sigma_R R(t) + \sigma_V V(t) - u(t)S(t) - \mu_S S(t) \\ \dot{I}(t) = \beta S(t)I(t-h) + \theta_V \beta V(t)I(t-h) + \theta_R \beta R(t)I(t-h) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \sigma_R R(t) - \theta_R \beta R(t)I(t-h) - \mu_R R(t) \\ \dot{V}(t) = u(t)S(t) - \sigma_V V(t) - \theta_V \beta V(t)I(t-h) - \mu_V V(t) \end{cases} \quad (7)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2} I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (7)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, I, R, V, S_h, I_h, R_h, V_h, u, \lambda_S, \lambda_I, \lambda_R, \lambda_V)$ is

$$\begin{aligned} \mathcal{H} = & w_I I + w_u u^2 + \lambda_S (\Lambda - \beta S I_h + \sigma_R R + \sigma_V V - u S - \mu_S S) \\ & + \lambda_I (\beta S I_h + \theta_V \beta V I_h + \theta_R \beta R I_h - \gamma I - \mu_I I) \\ & + \lambda_R (\gamma I - \sigma_R R - \theta_R \beta R I_h - \mu_R R) + \lambda_V (u S - \sigma_V V - \theta_V \beta V I_h - \mu_V V). \end{aligned}$$

Proposition 3.5 (Extended SIRV Optimality System). *Let $(S^*, I^*, R^*, V^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R, \lambda_V$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta I^*(t-h)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t) + u^*(t)(\lambda_V(t) - \lambda_S(t)), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I) \lambda_I(t) + \gamma \lambda_R(t) + \mathbf{1}_{[t_0, T-h]}(t) [\beta S^*(t+h)(\lambda_I(t+h) - \lambda_S(t+h)) \\ &\quad + \theta_V \beta V^*(t+h)(\lambda_I(t+h) - \lambda_V(t+h)) + \theta_R \beta R^*(t+h)(\lambda_I(t+h) - \lambda_R(t+h))] \\ -\dot{\lambda}_R(t) &= \sigma_R (\lambda_S(t) - \lambda_R(t)) + \theta_R \beta I^*(t-h)(\lambda_I(t) - \lambda_R(t)) - \mu_R \lambda_R(t), \\ -\dot{\lambda}_V(t) &= \sigma_V (\lambda_S(t) - \lambda_V(t)) + \theta_V \beta I^*(t-h)(\lambda_I(t) - \lambda_V(t)) - \mu_V \lambda_V(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0, \quad \lambda_V(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{S^*(t)(\lambda_S(t) - \lambda_V(t))}{2w_u} \right].$$

3.6 Extended SIRV Model with Double Delay

In the extended SIRV model, we consider now both the delay $h_1 > 0$ associated with the incubation period and the delay $h_2 > 0$ associated with the time required for the vaccine to confer protection. We consider the control variable $u \in [0, u_{\max}]$ that represents the vaccination rate. The resulting dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t - h_1) + \sigma_R R(t) + \sigma_V V(t) - u(t)S(t - h_2) - \mu_S S(t) \\ \dot{I}(t) = \beta S(t)I(t - h_1) + \theta_V \beta V(t)I(t - h_1) + \theta_R \beta R(t)I(t - h_1) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \sigma_R R(t) - \theta_R \beta R(t)I(t - h_1) - \mu_R R(t) \\ \dot{V}(t) = u(t)S(t - h_2) - \sigma_V V(t) - \theta_V \beta V(t)I(t - h_1) - \mu_V V(t) \end{cases} \quad (8)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{1}{2} I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (8)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, I, R, V, S_{h_1}, I_{h_1}, R_{h_1}, V_{h_1}, S_{h_2}, I_{h_2}, R_{h_2}, V_{h_2}, u, \lambda_S, \lambda_I, \lambda_R, \lambda_V)$ is

$$\begin{aligned} \mathcal{H} = & w_I I + w_u u^2 + \lambda_S (\Lambda - \beta S I_{h_1} + \sigma_R R + \sigma_V V - u S_{h_2} - \mu_S S) \\ & + \lambda_I (\beta S I_{h_1} + \theta_V \beta V I_{h_1} + \theta_R \beta R I_{h_1} - \gamma I - \mu_I I) \\ & + \lambda_R (\gamma I - \sigma_R R - \theta_R \beta R I_{h_1} - \mu_R R) + \lambda_V (u S_{h_2} - \sigma_V V - \theta_V \beta V I_{h_1} - \mu_V V). \end{aligned}$$

Proposition 3.6 (Extended SIRV Optimality System). *Let $(S^*, I^*, R^*, V^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R, \lambda_V$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta I^*(t - h_1)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t) + \mathbf{1}_{[t_0, T-h_2]}(t) u^*(t + h_2)(\lambda_V(t + h_2) - \lambda_S(t + h_2)), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I) \lambda_I(t) + \gamma \lambda_R(t) + \mathbf{1}_{[t_0, T-h_1]}(t) [\beta S^*(t + h_1)(\lambda_I(t + h_1) - \lambda_S(t + h_1)) \\ &\quad + \theta_V \beta V^*(t + h_1)(\lambda_I(t + h_1) - \lambda_V(t + h_1)) + \theta_R \beta R^*(t + h_1)(\lambda_I(t + h_1) - \lambda_R(t + h_1))] \\ -\dot{\lambda}_R(t) &= \sigma_R (\lambda_S(t) - \lambda_R(t)) + \theta_R \beta I^*(t - h_1)(\lambda_I(t) - \lambda_R(t)) - \mu_R \lambda_R(t), \\ -\dot{\lambda}_V(t) &= \sigma_V (\lambda_S(t) - \lambda_V(t)) + \theta_V \beta I^*(t - h_1)(\lambda_I(t) - \lambda_V(t)) - \mu_V \lambda_V(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0, \quad \lambda_V(T) = 0.$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{S^*(t - h_2)(\lambda_S(t) - \lambda_V(t))}{2w_u} \right].$$

3.7 Extended SIRV Model with Double Delay and Double Control

In the extended SIRV model, we consider now both the delay $h_1 > 0$ associated with the incubation period and the delay $h_2 > 0$ associated with the time required for the vaccine to confer protection. We also consider two control variables: $u \in [0, u_{\max}]$ represents the stringency of interventions, such as physical distancing or mask mandates, that reduce the transmission rate β , while $v \in [0, v_{\max}]$ represents the vaccination rate. The resulting dynamical system is

$$\begin{cases} \dot{S}(t) = \Lambda - \beta[1 - u(t)]S(t)I(t - h_1) + \sigma_R R(t) + \sigma_V V(t) - v(t)S(t - h_2) - \mu_S S(t) \\ \dot{I}(t) = \beta[1 - u(t)]S(t)I(t - h_1) + \theta_V \beta[1 - u(t)]V(t)I(t - h_1) \\ \quad + \theta_R \beta[1 - u(t)]R(t)I(t - h_1) - \gamma I(t) - \mu_I I(t) \\ \dot{R}(t) = \gamma I(t) - \sigma_R R(t) - \theta_R \beta[1 - u(t)]R(t)I(t - h_1) - \mu_R R(t) \\ \dot{V}(t) = v(t)S(t - h_2) - \sigma_V V(t) - \theta_V \beta[1 - u(t)]V(t)I(t - h_1) - \mu_V V(t) \end{cases} \quad (9)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot), v(\cdot)} \quad & \frac{1}{2} I(T)^2 + \int_{t_0}^T (w_I I(t) + w_u u(t)^2 + w_v v(t)^2) dt \\ \text{subject to} \quad & \text{system (9)} \\ & u(t) \in [0, u_{\max}] \\ & v(t) \in [0, v_{\max}] \end{aligned}$$

The Hamiltonian $\mathcal{H}(S, I, R, V, S_{h_1}, I_{h_1}, R_{h_1}, V_{h_1}, S_{h_2}, I_{h_2}, R_{h_2}, V_{h_2}, u, \lambda_S, \lambda_I, \lambda_R, \lambda_V)$ is

$$\begin{aligned} \mathcal{H} = & w_I I + w_u u^2 + w_v v^2 + \lambda_S (\Lambda - \beta(1 - u)SI_{h_1} + \sigma_R R + \sigma_V V - vS_{h_2} - \mu_S S) \\ & + \lambda_I (\beta(1 - u)SI_{h_1} + \theta_V \beta(1 - u)VI_{h_1} + \theta_R \beta(1 - u)RI_{h_1} - \gamma I - \mu_I I) \\ & + \lambda_R (\gamma I - \sigma_R R - \theta_R \beta(1 - u)RI_{h_1} - \mu_R R) + \lambda_V (vS_{h_2} - \sigma_V V - \theta_V \beta(1 - u)VI_{h_1} - \mu_V V). \end{aligned}$$

Proposition 3.7 (Extended SIRV Optimality System). *Let $(S^*, I^*, R^*, V^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_R, \lambda_V$ satisfying:*

Costate equations:

$$\begin{aligned} -\dot{\lambda}_S(t) &= \beta(1 - u^*(t))I^*(t - h_1)(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t) \\ &\quad + \mathbf{1}_{[t_0, T-h_2]}(t)v^*(t + h_2)(\lambda_V(t + h_2) - \lambda_S(t + h_2)), \\ -\dot{\lambda}_I(t) &= w_I - (\gamma + \mu_I)\lambda_I(t) + \gamma\lambda_R(t) \\ &\quad + \mathbf{1}_{[t_0, T-h_1]}(t)[\beta(1 - u^*(t + h_1))S^*(t + h_1)(\lambda_I(t + h_1) - \lambda_S(t + h_1)) \\ &\quad + \theta_V \beta(1 - u^*(t + h_1))V^*(t + h_1)(\lambda_I(t + h_1) - \lambda_V(t + h_1)) \\ &\quad + \theta_R \beta(1 - u^*(t + h_1))R^*(t + h_1)(\lambda_I(t + h_1) - \lambda_R(t + h_1))] \\ -\dot{\lambda}_R(t) &= \sigma_R(\lambda_S(t) - \lambda_R(t)) + \theta_R \beta(1 - u^*(t))I^*(t - h_1)(\lambda_I(t) - \lambda_R(t)) - \mu_R \lambda_R(t), \\ -\dot{\lambda}_V(t) &= \sigma_V(\lambda_S(t) - \lambda_V(t)) + \theta_V \beta(1 - u^*(t))I^*(t - h_1)(\lambda_I(t) - \lambda_V(t)) - \mu_V \lambda_V(t). \end{aligned}$$

Terminal conditions:

$$\lambda_S(T) = 0, \quad \lambda_I(T) = I^*(T), \quad \lambda_R(T) = 0, \quad \lambda_V(T) = 0.$$

Optimality condition:

$$\begin{aligned} u^*(t) &= P_{[0, u_{\max}]} \left[\frac{\beta S^*(t)I^*(t-h_1) [(\lambda_I(t) - \lambda_S(t)) + \theta_R(\lambda_I(t) - \lambda_R(t)) + \theta_V(\lambda_I(t) - \lambda_V(t))]}{2w_u} \right] \\ v^*(t) &= P_{[0, v_{\max}]} \left[\frac{S^*(t-h_2)(\lambda_S(t) - \lambda_V(t))}{2w_v} \right]. \end{aligned}$$

3.8 SIDARTHE-V Model with Double Delay

In the model in [9], with five infected compartments (I, D, A, R, T) , we introduce waning immunity with rates σ_i and (re)infection probabilities $\theta_i \in [0, 1]$ for recovered and vaccinated, as well as a birth rate Λ and non-COVID-related mortality rates μ_i , and we denote by v the vaccination rate [10]. We consider both the delay $h_1 > 0$ associated with the incubation period and the delay $h_2 > 0$ associated with the time required for the vaccine to confer protection. We adopt the notation $I_{h_1} := I(t - h_1)$ and $S_{h_2} := S(t - h_2)$.

$$\begin{cases} \dot{S} &= \Lambda - S(\alpha I_{h_1} + \beta D_{h_1} + \gamma A_{h_1} + \delta R_{h_1}) - u S_{h_2} + \sigma_H H + \sigma_V V - \mu_S S \\ \dot{I} &= S(\alpha I_{h_1} + \beta D_{h_1} + \gamma A_{h_1} + \delta R_{h_1}) - (\varepsilon + \zeta + \lambda)I + \theta_1 \alpha H I_{h_1} \\ &\quad + \theta_2 \alpha V I_{h_1} + \theta_3 \beta H D_{h_1} + \theta_4 \beta V D_{h_1} + \theta_5 \gamma H A_{h_1} + \theta_6 \gamma V A_{h_1} + \theta_7 \delta H R_{h_1} + \theta_8 \delta V R_{h_1} - \mu_I I \\ \dot{D} &= \varepsilon I - (\eta + \rho)D - \mu_D D \\ \dot{A} &= \zeta I - (\theta + \mu + \kappa)A - \mu_A A \\ \dot{R} &= \eta D + \theta A - (\nu + \xi + \tau_1)R - \mu_R R \\ \dot{T} &= \mu A + \nu R - (\sigma + \tau_2)T - \mu_T T \\ \dot{H} &= \lambda I + \rho D + \kappa A + \xi R + \sigma T - \sigma_H H - \theta_1 \alpha H I_{h_1} \\ &\quad - \theta_3 \beta H D_{h_1} - \theta_5 \gamma H A_{h_1} - \theta_7 \delta H R_{h_1} - \mu_H H \\ \dot{E} &= \tau_1 R + \tau_2 T \\ \dot{V} &= u S_{h_2} - \sigma_V V - \theta_2 \alpha V I_{h_1} - \theta_4 \beta V D_{h_1} - \theta_6 \gamma V A_{h_1} - \theta_8 \delta V R_{h_1} - \mu_V V \end{cases} \quad (10)$$

We wish to solve the optimal control problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \frac{I(T_H)^2 + D(T_H)^2 + A(T_H)^2 + R(T_H)^2 + T(T_H)^2}{2} \\ & + \int_{t_0}^{T_H} (w_I I(t) + w_D D(t) + w_A A(t) + w_R R(t) + w_T T(t) + w_u u(t)^2) dt \\ \text{subject to} \quad & \text{system (10)} \\ & u(t) \in [0, u_{\max}] \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H} = & w_I I + w_D D + w_A A + w_R R + w_T T + w_u u^2 \\ & + \lambda_S (\Lambda - S(\alpha I_{h_1} + \beta D_{h_1} + \gamma A_{h_1} + \delta R_{h_1}) - u S_{h_2} + \sigma_H H + \sigma_V V - \mu_S S) \\ & + \lambda_I (S(\alpha I_{h_1} + \beta D_{h_1} + \gamma A_{h_1} + \delta R_{h_1}) - (\varepsilon + \zeta + \lambda)I + \theta_1 \alpha H I_{h_1} + \theta_2 \alpha V I_{h_1} \\ & \quad + \theta_3 \beta H D_{h_1} + \theta_4 \beta V D_{h_1} + \theta_5 \gamma H A_{h_1} + \theta_6 \gamma V A_{h_1} + \theta_7 \delta H R_{h_1} + \theta_8 \delta V R_{h_1} - \mu_I I) \\ & + \lambda_D (\varepsilon I - (\eta + \rho)D - \mu_D D) \\ & + \lambda_A (\zeta I - (\theta + \mu + \kappa)A - \mu_A A) \\ & + \lambda_R (\eta D + \theta A - (\nu + \xi + \tau_1)R - \mu_R R) \\ & + \lambda_T (\mu A + \nu R - (\sigma + \tau_2)T - \mu_T T) \\ & + \lambda_H (\lambda I + \rho D + \kappa A + \xi R + \sigma T - \sigma_H H - \theta_1 \alpha H I_{h_1} - \theta_3 \beta H D_{h_1} - \theta_5 \gamma H A_{h_1} - \theta_7 \delta H R_{h_1} - \mu_H H) \\ & + \lambda_E (\tau_1 R + \tau_2 T) \\ & + \lambda_V (u S_{h_2} - \sigma_V V - \theta_2 \alpha V I_{h_1} - \theta_4 \beta V D_{h_1} - \theta_6 \gamma V A_{h_1} - \theta_8 \delta V R_{h_1} - \mu_V V). \end{aligned}$$

Proposition 3.8 (SIDARTHE-V Optimality System). *Let $(S^*, I^*, D^*, A^*, R^*, T^*, H^*, E^*, V^*, u^*)$ be an optimal solution. Then there exist costate variables $\lambda_S, \lambda_I, \lambda_D, \lambda_A, \lambda_R, \lambda_T, \lambda_H, \lambda_E, \lambda_V$ satisfying:*

Costate equations:

$$\begin{aligned}
-\dot{\lambda}_S(t) &= (\alpha I^*(t - h_1) + \beta D^*(t - h_1) + \gamma A^*(t - h_1) + \delta R^*(t - h_1))(\lambda_I(t) - \lambda_S(t)) - \mu_S \lambda_S(t) \\
&\quad + \mathbf{1}_{[t_0, T_H - h_2]}(t) u^*(t + h_2)(\lambda_V(t + h_2) - \lambda_S(t + h_2)), \\
-\dot{\lambda}_I(t) &= w_I - (\varepsilon + \zeta + \lambda + \mu_I) \lambda_I(t) + \varepsilon \lambda_D(t) + \zeta \lambda_A(t) + \lambda \lambda_H(t) \\
&\quad + \mathbf{1}_{[t_0, T_H - h_1]}(t) \alpha (S^*(t)(\lambda_I(t) - \lambda_S(t)) + \theta_1 H^*(t)(\lambda_I(t) - \lambda_H(t)) + \theta_2 V^*(t)(\lambda_I(t) - \lambda_V(t))) \\
-\dot{\lambda}_D(t) &= w_D - (\eta + \rho + \mu_D) \lambda_D(t) + \eta \lambda_R(t) + \rho \lambda_H(t) \\
&\quad + \mathbf{1}_{[t_0, T_H - h_1]}(t) \beta (S^*(t)(\lambda_I(t) - \lambda_S(t)) + \theta_3 H^*(t)(\lambda_I(t) - \lambda_H(t)) + \theta_4 V^*(t)(\lambda_I(t) - \lambda_V(t))) \\
-\dot{\lambda}_A(t) &= w_A - (\theta + \mu + \kappa + \mu_A) \lambda_A(t) + \theta \lambda_R(t) + \mu \lambda_T(t) + \kappa \lambda_H(t) \\
&\quad + \mathbf{1}_{[t_0, T_H - h_1]}(t) \gamma (S^*(t)(\lambda_I(t) - \lambda_S(t)) + \theta_5 H^*(t)(\lambda_I(t) - \lambda_H(t)) + \theta_6 V^*(t)(\lambda_I(t) - \lambda_V(t))) \\
-\dot{\lambda}_R(t) &= w_R - (\nu + \xi + \tau_1 + \mu_R) \lambda_R(t) + \nu \lambda_T(t) + \xi \lambda_H(t) + \tau_1 \lambda_E(t) \\
&\quad + \mathbf{1}_{[t_0, T_H - h_1]}(t) \delta (S^*(t)(\lambda_I(t) - \lambda_S(t)) + \theta_7 H^*(t)(\lambda_I(t) - \lambda_H(t)) + \theta_8 V^*(t)(\lambda_I(t) - \lambda_V(t))) \\
-\dot{\lambda}_T(t) &= w_T - (\sigma + \tau_2 + \mu_T) \lambda_T(t) + \sigma \lambda_H(t) + \tau_2 \lambda_E(t) \\
-\dot{\lambda}_H(t) &= \sigma_H (\lambda_S(t) - \lambda_H(t)) - \mu_H \lambda_H(t) \\
&\quad + (\theta_1 \alpha I^*(t - h_1) + \theta_3 \beta D^*(t - h_1) + \theta_5 \gamma A^*(t - h_1) + \theta_7 \delta R^*(t - h_1))(\lambda_I(t) - \lambda_H(t)) \\
-\dot{\lambda}_E(t) &= 0 \\
-\dot{\lambda}_V(t) &= \sigma_V (\lambda_S(t) - \lambda_V(t)) - \mu_V \lambda_V(t) \\
&\quad + (\theta_2 \alpha I^*(t - h_1) + \theta_4 \beta D^*(t - h_1) + \theta_6 \gamma A^*(t - h_1) + \theta_8 \delta R^*(t - h_1))(\lambda_I(t) - \lambda_V(t)) \\
-\dot{\lambda}_R(t) &= \sigma_R (\lambda_S(t) - \lambda_R(t)) + \theta_R \beta (1 - u^*(t)) I^*(t - h) (\lambda_I(t) - \lambda_R(t)) - \mu_R \lambda_R(t), \\
-\dot{\lambda}_V(t) &= \sigma_V (\lambda_S(t) - \lambda_V(t)) + \theta_V \beta (1 - u^*(t)) I^*(t - h) (\lambda_I(t) - \lambda_V(t)) - \mu_V \lambda_V(t).
\end{aligned}$$

Terminal conditions:

$$\begin{aligned}
\lambda_S(T_H) &= 0, \quad \lambda_I(T_H) = I^*(T_H), \quad \lambda_D(T_H) = D^*(T_H), \quad \lambda_A(T_H) = A^*(T_H), \\
\lambda_R(T_H) &= R^*(T_H), \quad \lambda_T(T_H) = T^*(T_H), \quad \lambda_H(T_H) = 0, \quad \lambda_E(T_H) = 0, \quad \lambda_V(T_H) = 0.
\end{aligned}$$

Optimality condition:

$$u^*(t) = P_{[0, u_{\max}]} \left[\frac{S^*(t - h_2)(\lambda_S(t) - \lambda_V(t))}{2w_u} \right].$$

4 Numerical simulations

We perform a campaign of numerical experiments by solving optimal control problems (OCPs) with discrete delays, of the form (1), subject to the dynamics of the epidemiological models introduced in Section 3, using our proposed extended Sakawa-Shindo algorithm (ESSA).

A key preliminary step is the construction of a time grid that is compatible with all delays present in the problem. In the case of single delay, a uniform grid with a fixed time step is adopted. When multiple delays $h_1, h_2, \dots$ are present, each one imposes a natural mesh refinement scale. Rather than choosing a single uniform step, a non-uniform composite grid is assembled by merging the grids induced by each delay and removing duplicate nodes. This ensures that each delay value corresponds exactly to an integer number of steps, so that delayed quantities can be retrieved by direct index lookup.

An extended grid is prepended to cover the history interval $[t_0 - h, t_0]$, where h is the largest delay. Over this interval, the state is prescribed by the initial history function, which in all problems considered here is taken to be the constant initial condition.

The forward state equations are solved through an explicit Euler discretization method, while the backward costate integration is performed with an implicit backward Euler formula.

Once the costate is available, new candidate controls are computed pointwise by minimising the Hamiltonian with respect to each control variable, subject to the box constraints. The updated controls are then used to propagate the state forward using a predictor–corrector scheme, as for the original Sakawa-Shindo algorithm [8]: a forward Euler step provides a first estimate of the next state (the predictor), which is then corrected by a trapezoidal rule step (the corrector), achieving second-order accuracy in time.

4.1 SIR Model with Delay

We first consider the SIR model with delay described in Section 3.1. The epidemiological parameters and initial conditions are reported in Figure 1. The four panels show the optimal trajectories of the state variables $S(t)$, $I(t)$, $R(t)$ and the optimal control $u(t)$ for increasing values of the delay: $h = 1$ (top, left), $h = 5$ (top, right), $h = 7$ (bottom, left) and $h = 15$ (bottom, right).

In all cases, the optimal control takes its maximum admissible value $u_{\max} = 0.8$ over an initial interval and then drops to zero once the infected population size has sufficiently decreased.

The most notable feature is the relation between the delay h and the duration of the optimal intervention. As h increases from 1 to 15 days, the period over which the control remains at its maximum value lengthens substantially: for $h = 1$ the intervention is brief and concentrated in the early phase, while for $h = 15$ the control remains active until approximately day 220. This behaviour has a transparent epidemiological interpretation: a longer incubation period (corresponding to a larger delay) means that newly generated infections take more time to become visible in the I compartment (and hence spread the contagion), so the system responds more slowly to any applied control. As a consequence, a longer and more sustained intervention is required to achieve the same degree of epidemic suppression. Equivalently, the controller must act for a longer time to compensate for the fact that the effect of its actions is only felt after a delay of h days. This result underscores the importance of accounting for realistic incubation delays when designing and evaluating public health interventions: models that neglect the delay will systematically underestimate the duration of the measures needed to contain an outbreak.

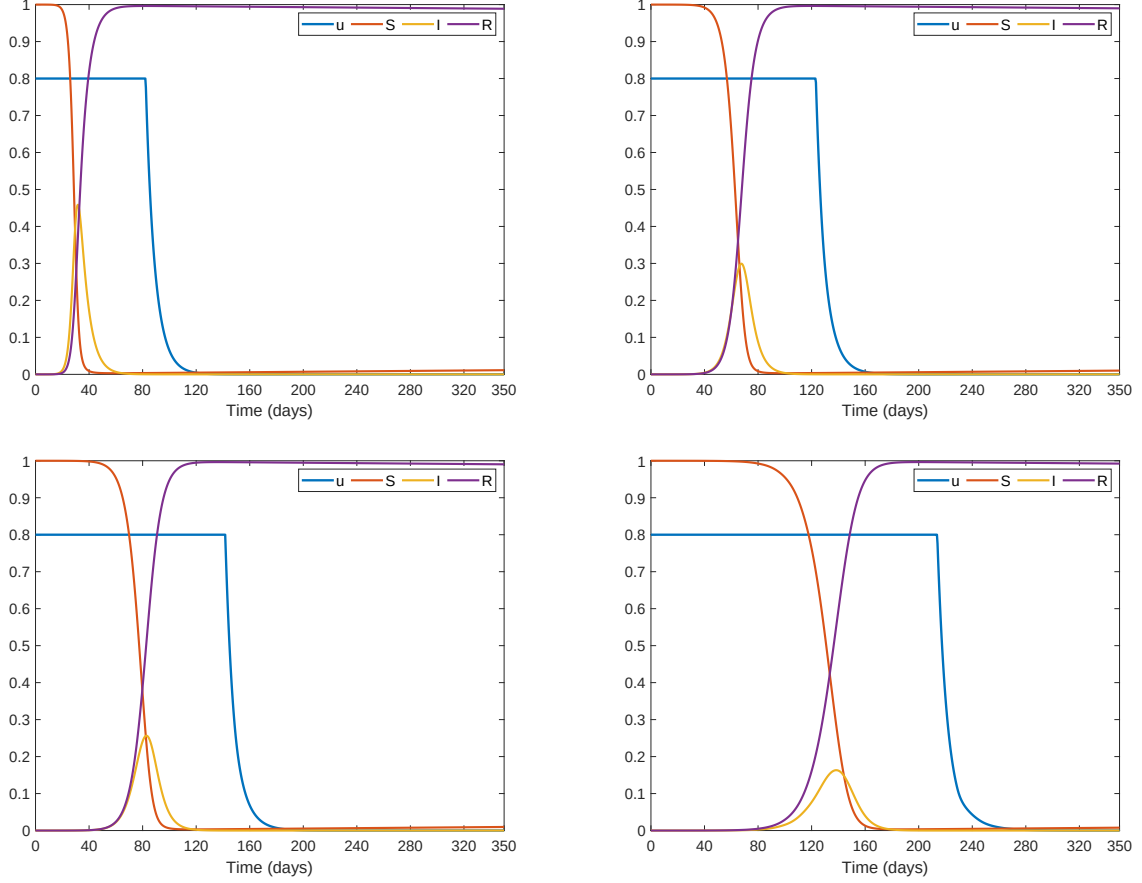


Figure 1: SIR OCP with delay and a high transmission rate $\beta = 5$, with the other parameters chosen as in [10], i.e., $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i , $\gamma = 1/6$. The cost weights are $w_I = 10^5$, $w_u = 1$, the control bound is $u_{\max} = 0.8$, and the initial conditions are $I_0 = 10^{-6}$, $R_0 = 0$, $S_0 = 1 - I_0$. The considered delays are: $h = 1$ (top, left), $h = 5$ (top, right), $h = 7$ (bottom, left), $h = 15$ (bottom, right).

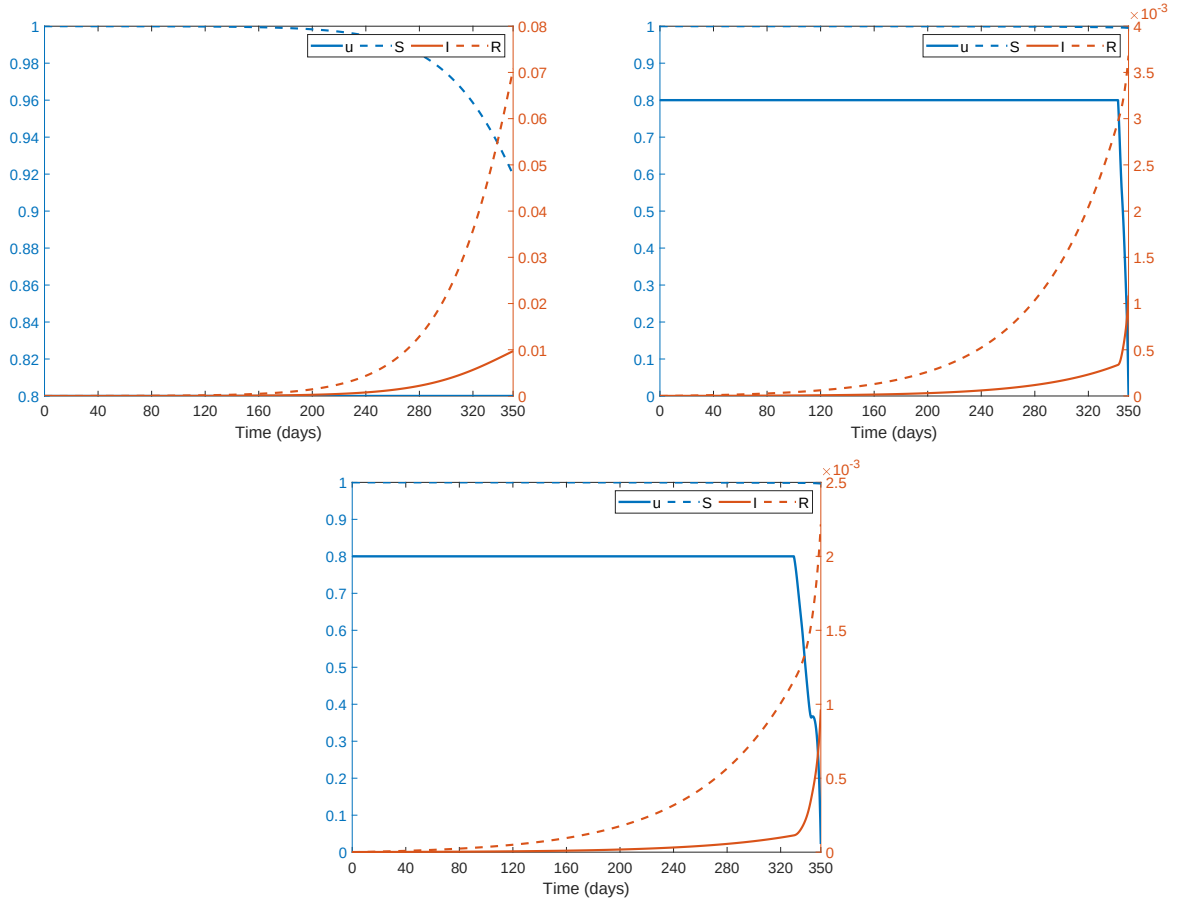


Figure 2: SIRS OCP with delay, with parameters chosen as in [10], i.e., $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i , $\beta = 1$, $\gamma = 1/6$ and $\delta = 1/500$. The cost weights are $w_I = 10^3$, $w_u = 1$, the control bound is $u_{\max} = 0.8$, and the initial conditions are $I_0 = 10^{-6}$, $R_0 = 0$, $S_0 = 1 - I_0$. The consider delays are: $h = 1$ (top left), $h = 5$ (top right), $h = 7$ (bottom).

4.2 SIRS Model with Delay

We consider the SIRS model described in Section 3.2. The parameters used in this numerical simulation differ from the SIR case in two respects: the transmission coefficient is reduced to $\beta = 1$, reflecting a less aggressive pathogen, and thus also the cost weight on the control effort is decreased to $w_u = 10^2$. All the parameters are as reported in the caption of Figure 2.

The three panels show the optimal state trajectories and control for $h = 1$ (top left), $h = 5$ (top right), $h = 7$ (bottom). Also in this case, a larger delay requires a longer and more sustained intervention to achieve a comparable degree of epidemic suppression.

We can notice that the size of the infected population is markedly smaller than in Figure 1 across all three panels, reflecting the lower transmission rate $\beta = 1$.

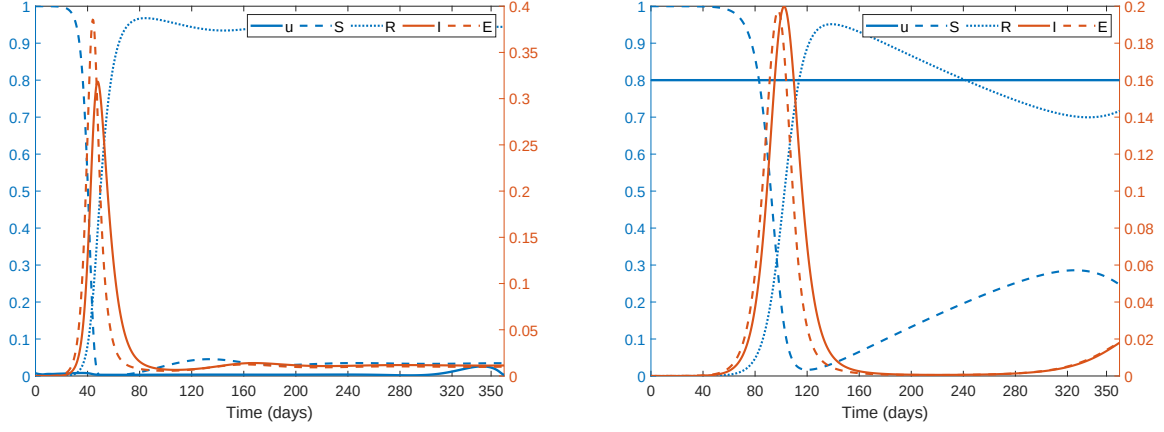


Figure 3: SEIRS OCP with delay $h = 5$, with $\alpha = 0.19$, $\beta = 5$, $\gamma = 1/6$ and $\delta = 1/500$, $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i . The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $E_0 = 0$, $R_0 = 0$. The cost functional weights are: $w_I = w_E = w_u = 1$ (left), $w_I = w_E = 10^5$ and $w_u = 1$ (right).

4.3 SEIRS Model with Delay

For the SEIRS model described in Section 3.3, in Figure 3 we consider a delay of $h = 5$ days and set the cost functional weights $w_I = w_E = w_u = 1$ (left) and $w_I = w_E = 10^5$ and $w_u = 1$ (right). In the first case, the control values are extremely small, whereas in the second case the control is kept at its maximum value, thus significantly reducing the peak of infections; however, it is unable to suppress the epidemic entirely, due to the high transmission rate $\beta = 5$.

Next, fixing the cost weights at $w_I = w_E = 10^5$ and $w_u = 1$, we investigate the effect of increasing the delay h ; the corresponding results are shown in Figure 4. In all cases, the control is kept at its maximum value; the presence of the delay causes multiple epidemic waves, while increasing the delay shifts the epidemic peaks later in time, and reduces the peak amplitude.

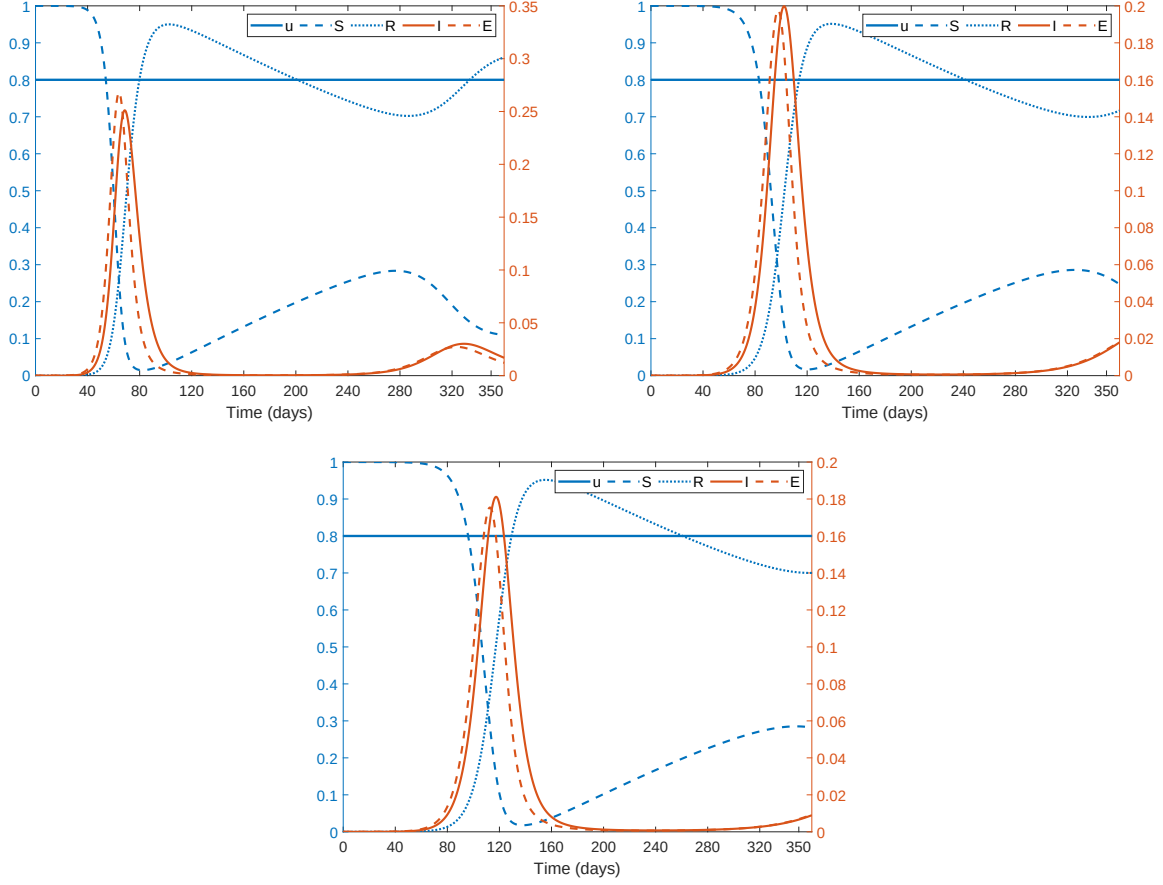


Figure 4: SEIRS OCP with delay, with $\alpha = 0.19$, $\beta = 5$, $\gamma = 1/6$ and $\delta = 1/500$, $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i . The control bound is $u_{\max} = 0.8$. The cost functional weights are $w_I = w_E = 10^5$, $w_u = 1$ and the initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $E_0 = 0$, $R_0 = 0$. The delay is: $h = 1$ (top left), $h = 5$ (top right), $h = 7$ (bottom).

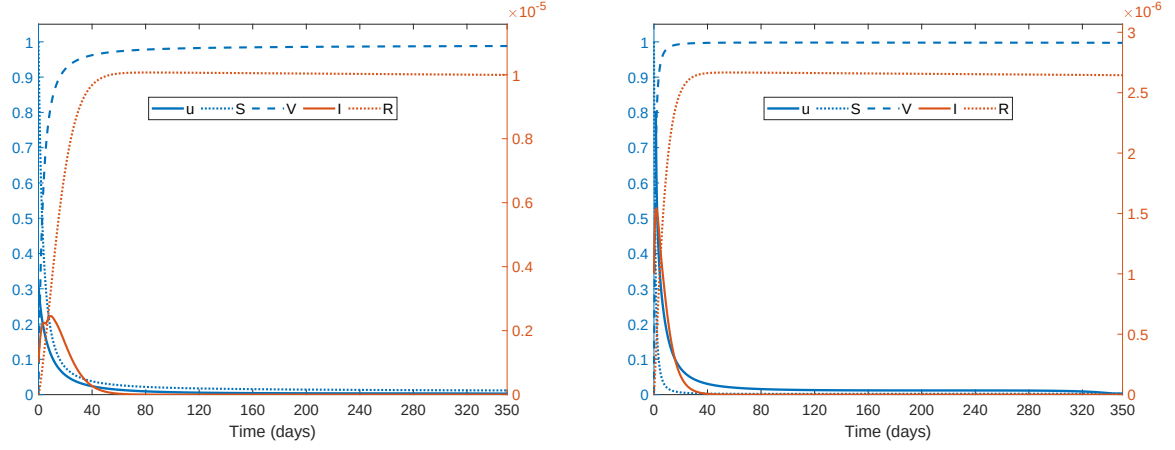


Figure 5: SIRV OCP with delay, with parameters chosen as in [10], i.e., $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i , $\beta = 1$ and $\gamma = 1/6$. The delay is $h = 5$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^2$ and $w_u = 1$ (right).

4.4 SIRV Model with Delay

For the SIRV model described in Section 3.4, we consider three values of the delay parameter: $h = 3$ in Figure 6, $h = 5$ in Figure 5, and $h = 15$ in Figure 5. In each case, we consider two cost weight configurations: a balanced regime with $w_I = w_u = 1$, and an infection-penalizing regime with $w_I = 10^2$ and $w_u = 1$.

Across all experiments, the optimal control (vaccination rate) takes large values in the early phase of the epidemic and then decreases. This behaviour is consistent with the intuition that the vaccination effort is most valuable before the infectious compartments I reaches its peak.

Comparing the left and right panels within each figure, the effect of increasing w_I from 1 to 10^2 is clearly visible. When the number of infections is penalized more heavily, a larger vaccination effort is allocated early in the time horizon, resulting in a faster reduction in the number of susceptible individuals and a further reduction in the number of infections. The qualitative shape of the control and state trajectories is preserved, but the transition is sharper and the peak of infection occurs at lower values, as expected due to the increased cost associated with infections.

For small delay, as shown in Figure 6 for the case of $h = 3$ days, the vaccination campaign builds up smoothly, keeping the number of infections very low throughout the simulated horizon. The number of susceptible individuals decreases in a nearly monotone fashion, while the number of vaccinated individual increases in proportion, and the state trajectories are qualitatively similar to the delay-free case.

As the delay increases to $h = 5$ days, as shown in Figure 5, the transient is slower, though the overall epidemic trajectory remains well controlled.

For the largest delay considered, $h = 15$ days, as shown in Figure 7, the number of susceptible individuals decreases more gradually in the early phase, the peak of infections is larger and the transient is slower, and the control values remain higher than in the other cases for a longer time period.

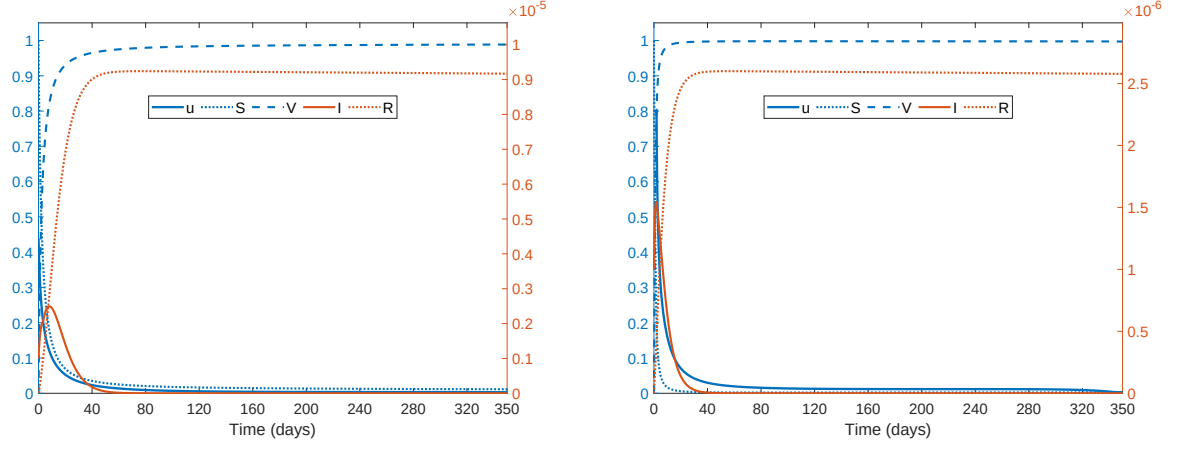


Figure 6: SIRV OCP with delay, with parameters chosen as in [10], i.e., $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i , $\beta = 1$ and $\gamma = 1/6$. The delay is $h = 3$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^2$ and $w_u = 1$ (right).

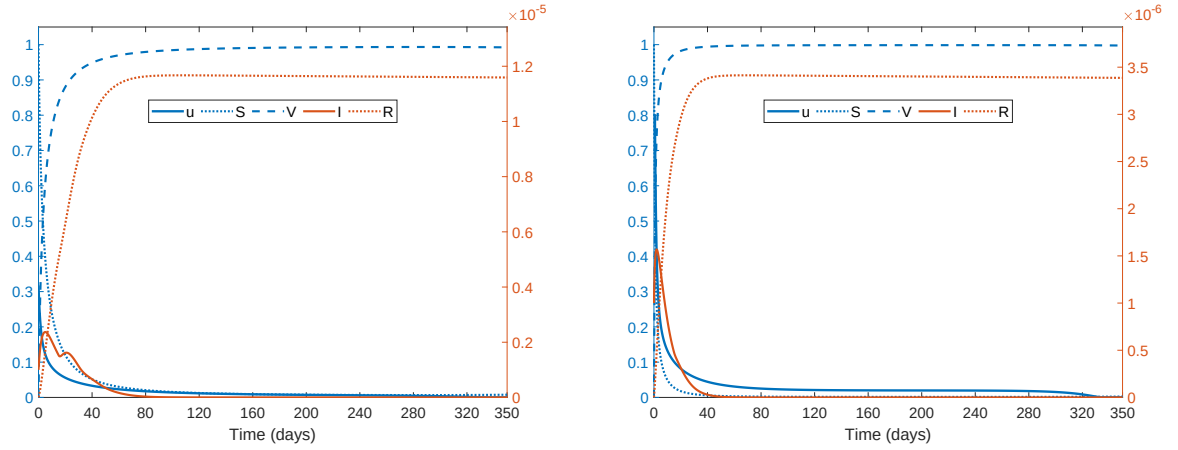


Figure 7: SIRV OCP with delay, with parameters chosen as in [10], i.e., $T = 350$ days, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_i = 2.90 \cdot 10^{-5}$ for all i , $\beta = 1$ and $\gamma = 1/6$. The delay is $h = 15$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^2$ and $w_u = 1$ (right).

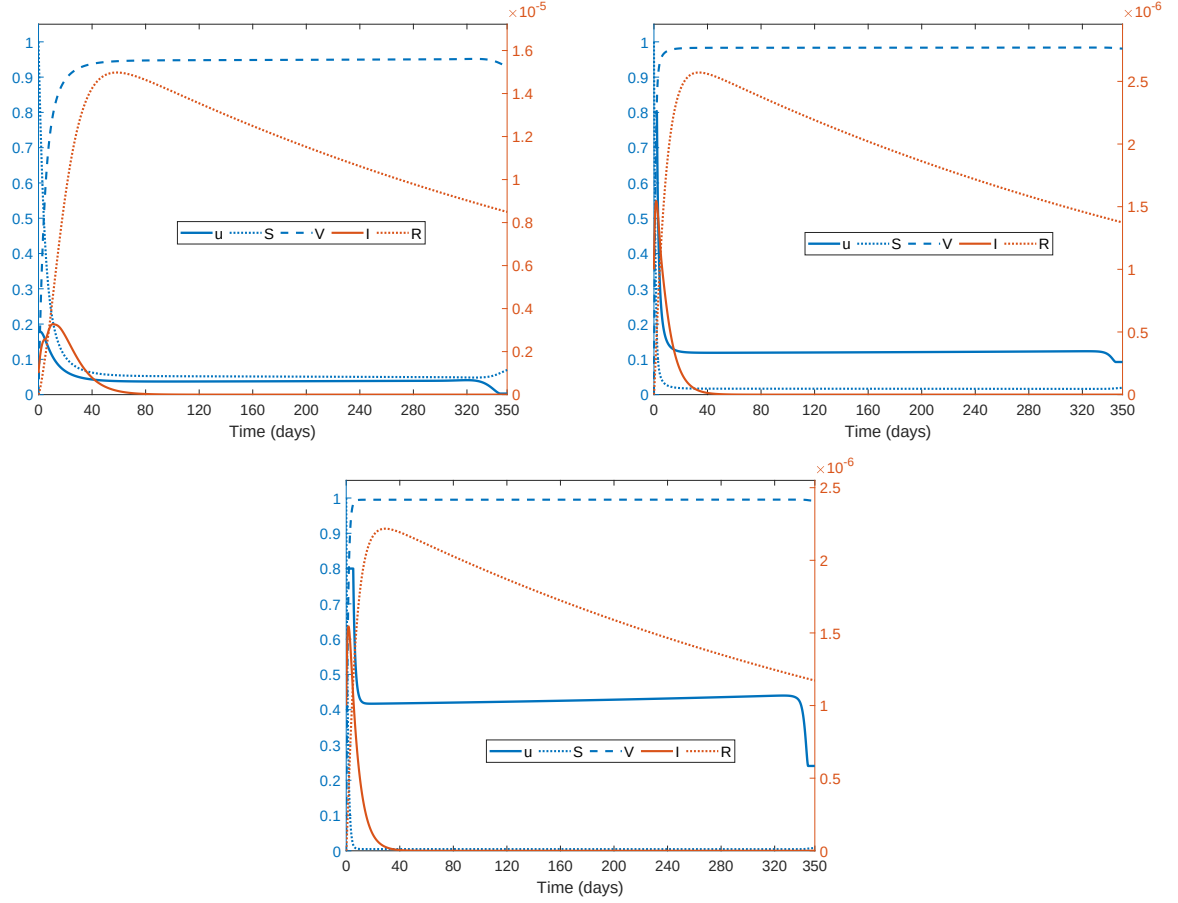


Figure 8: Extended SIRV OCP with delay, with the same parameters as for the SIRV model (see, e.g., Figure 5) and $\sigma_V = 1/500$, $\sigma_R = 1/500$, $\theta_V = 0.0013$, $\theta_R = 0.0021$. The delay is $h = 5$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (top left), $w_I = 10^2$ and $w_u = 1$ (top right), $w_I = 10^4$ and $w_u = 1$ (bottom).

4.5 Extended SIRV Model with Delay

Also for the extended SIRV model described in Section 3.5, we consider three values of the delay parameter: $h = 3$ in Figure 9, $h = 5$ in Figure 8, and $h = 15$ in Figure 10. In each case, we consider different cost weights. As in the SIRV case, increasing the cost weights on the number of infections leads to a higher vaccination rate, which populates faster the vaccinated compartment and depletes the susceptible compartment in proportion, thus also reducing the peak of infections. As the time delay increases, the peak of infections reduces, but the bell shape in the infected curve becomes wider. The effect of waning and imperfect immunity is visible in the need for a sustained vaccination campaign (with higher control values) throughout the simulated horizon.

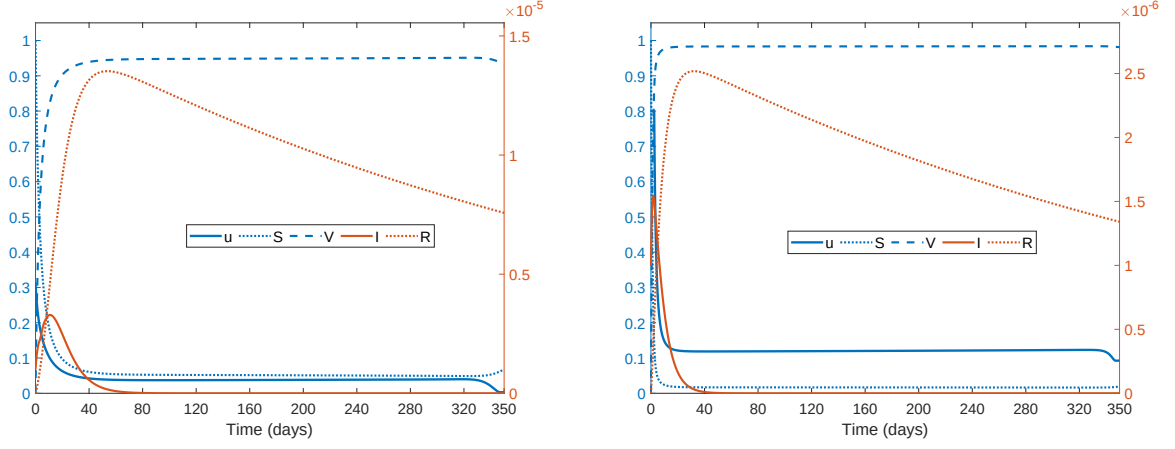


Figure 9: Extended SIRV OCP with delay, with the same parameters as for the SIRV model (see, e.g., Figure 5) and $\sigma_V = 1/500$, $\sigma_R = 1/500$, $\theta_V = 0.0013$, $\theta_R = 0.0021$. The delay is $h = 3$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^2$ and $w_u = 1$ (right).

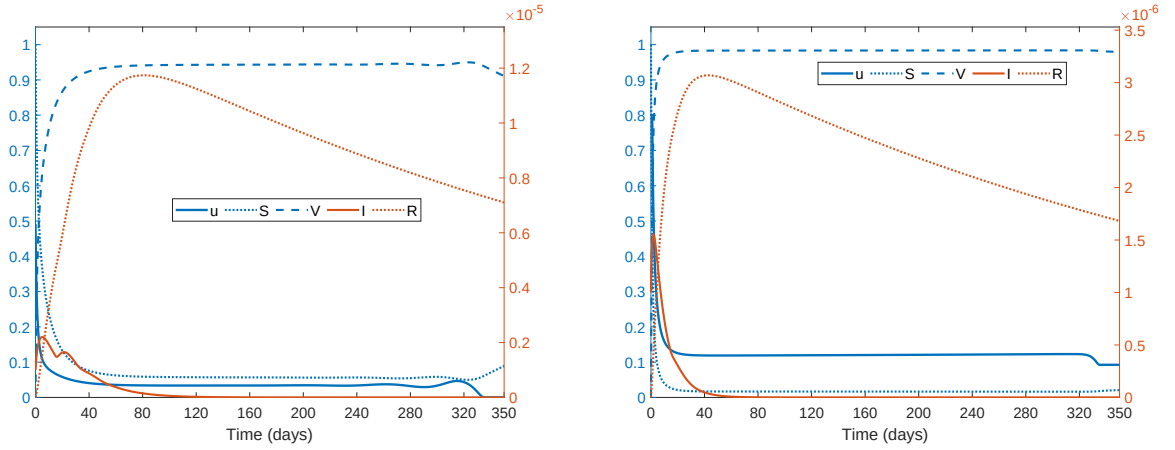


Figure 10: Extended SIRV OCP with delay, with the same parameters as for the SIRV model (see, e.g., Figure 5) and $\sigma_V = 1/500$, $\sigma_R = 1/500$, $\theta_V = 0.0013$, $\theta_R = 0.0021$. The delay is $h = 15$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^2$ and $w_u = 1$ (right).

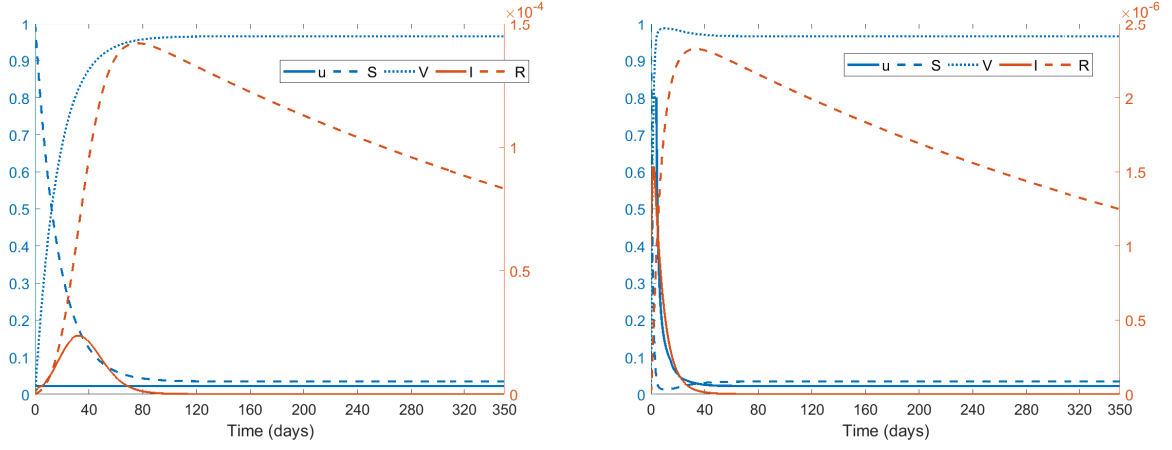


Figure 11: Extended SIRV OCP with double delay, with parameters as for the extended SIRV model (see, e.g., Figure 8) and delays $h_1 = 5$ days and $h_2 = 7$ days. The control bound is $u_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 1$ and $w_u = 1$ (left), $w_I = 10^5$ and $w_u = 1$ (right).

4.6 Extended SIRV Model with Double Delay

We now consider the extended SIRV model with a double delay described in Section 3.6. In particular, we set $h_1 = 5$ days for the delay related to the incubation period and $h_2 = 7$ days for the delay related to the time required for the vaccine to confer protection. The numerical results are shown in Figure 11. The introduction of the delay h_2 in the build-up of vaccine efficacy significantly alters the qualitative behaviour of the solution. The peak of $I(t)$ is of one or two orders of magnitude higher compared to the single-delay case. This is a consequence of the informational lag introduced by h_2 : the controller must decide how much to vaccinate at time t knowing that the protective effect will only materialize at time $t + h_2$. This anticipation requirement forces the optimal policy to act more conservatively and less reactively, since aggressive early vaccination does not prevent infections in the immediate window $[t, t + h_2]$. However, as expected, increasing the cost functional weight w_I results in administering more vaccines and reducing the peak of infections.

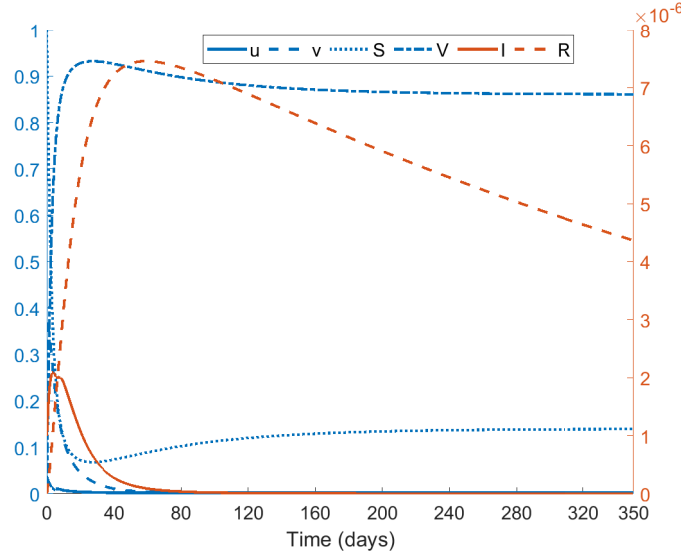


Figure 12: Extended SIRV OCP with double delay and double control, with parameters as for the extended SIRV model (see, e.g., Figure 8), delays $h_1 = 5$ days and $h_2 = 7$ days, and control bounds $u_{\max} = 0.4$ and $v_{\max} = 0.8$. The initial conditions are $I_0 = 10^{-6}$, $S_0 = 1 - I_0$, $R_0 = 0$, $V_0 = 0$. The cost functional weights are $w_I = 10^4$, $w_u = 1$, $w_v = 1$.

4.7 Extended SIRV Model with Double Delay and Double Control

Finally, we consider the extended SIRV model with double delay and double control described in Section 3.7, with delays $h_1 = 5$ days and $h_2 = 7$ days, and control bounds $u_{\max} = 0.4$ and $v_{\max} = 0.8$. The results for several cost weights configurations are illustrated in Figure 12 and Figure 13.

When $w_I = 10^4$ and $w_u = 1$, $w_v = 1$, as shown in Figure 12, the infected compartment displays two consecutive peaks of approximately $2.3 \cdot 10^{-6}$ within the first 20 days. In this balanced-cost regime, vaccination is favoured over non-pharmaceutical interventions, and the combined action of both controls is sufficient to suppress the epidemic within approximately 80 days. We then progressively increase the penalty on the vaccination control v . Specifically, in Figure 13, we set $w_u = 1$, $w_v = 10$ (top) and $w_u = 1$, $w_v = 10^2$ (centre). As a result, the algorithm reallocates the effort toward the non-pharmaceutical control u , as the higher cost of the vaccination v discourages its use. This reallocation comes at an epidemiological cost: the peak of infections increases by one order of magnitude, and its occurrence is delayed to approximately 20 and 35 days, reflecting the reduced effectiveness of isolation alone in suppressing early transmission under the double-delay structure.

Conversely, in the bottom panels of Figure 13, the penalty on the isolation control u is instead increased. For $w_u = 10$, $w_v = 1$ (left) and $w_u = 10^2$, $w_v = 1$ (right), the optimal policy shifts toward vaccination, which proves more effective: the peaks of infected are lower and the epidemic is contained within the first 80 days in both cases, underscoring the comparative advantage of vaccination (whenever available) over non-pharmaceutical interventions in this delayed SIRV framework.

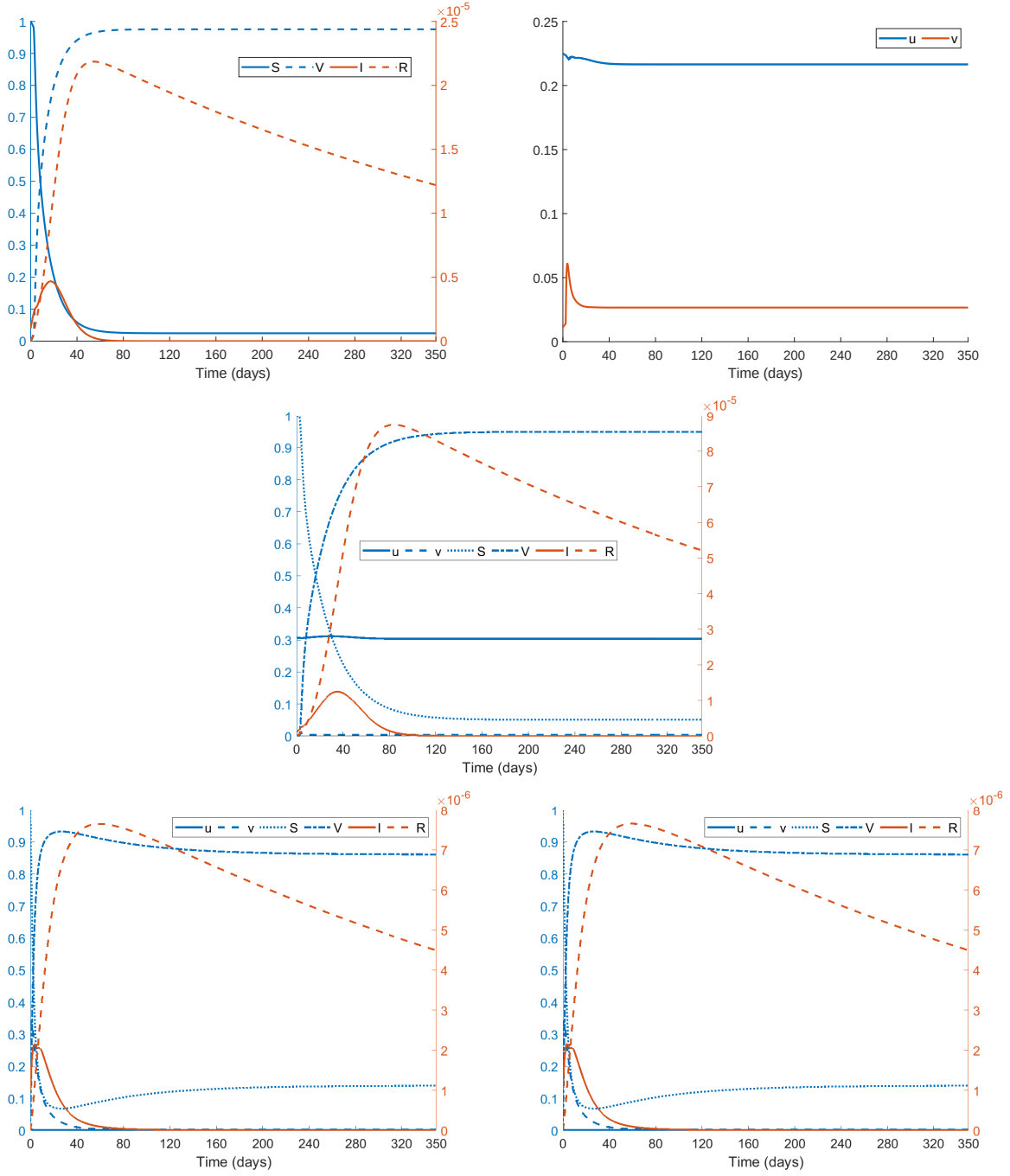


Figure 13: Extended SIRV OCP with double delay and double control, with the same parameters, control bounds and initial conditions as in Figure 12. The cost functional weights are $w_I = 10^4$, $w_u = 1$, $w_v = 10$ (top, left and right) and $w_I = 10^4$, $w_u = 1$, $w_v = 10^2$ (centre), $w_I = 10^4$, $w_u = 10$, $w_v = 1$ (bottom left), $w_I = 10^4$, $w_u = 10^2$, $w_v = 1$ (bottom right).

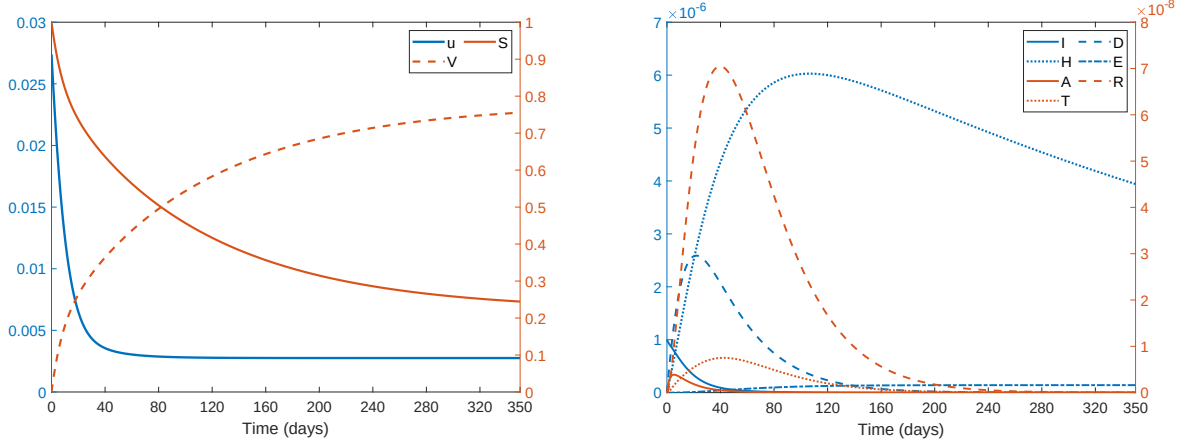


Figure 14: Extended SIDARTHE-V OCP with double delay, with parameters as in [10], i.e., $u_{\max} = 0.1$, $\Lambda = 2.91 \cdot 10^{-5}$, $\mu_I = 6.25 \cdot 10^5$, $\mu_j = 2.9 \cdot 10^{-5}$ for $j \neq I$, $\alpha = 0.3567$, $\beta = 0.0053$, $\gamma = 0.1485$, $\delta = 0.005$, $\epsilon = 0.2988$, $\zeta = 0.0025$, $\lambda = 0.1128$, $\eta = 0.0015$, $\rho = 0.032$, $\theta = 0.37$, $\mu = 0.12$, $\kappa = 0.02$, $\tau_1 = 0.005$, $\tau_2 = 0.17$, $\nu = 0.02$, $\xi = 0.022$, $\sigma = 0.024$, $\sigma_H = 1/500$, $\sigma_V = 1/500$, $\theta_i = 0.0013$, $i = 1, 3, 4, \dots, 8$, and $\theta_2 = 0.0021$. Initial conditions are $I_0 = 10^{-6}$, $D_0 = A_0 = R_0 = T_0 = H_0 = E_0 = V_0 = 0$, $S_0 = 1 - I_0$. The delays are $h_1 = 3$ days and $h_2 = 5$ days. The cost functional weights are $w_i = 10^3$, $i = I, D, A, R, T$, and $w_u = 1$.

4.8 SIDARTHE-V Model with Double Delay

We now consider the most complex model studied in this work, the extended SIDARTHE-V model with double delay described in Section 3.8, with a single transmission-reduction control $u(t) \in [0, u_{\max}]$, with $u_{\max} = 0.1$ as in [10]. The two delays reflect, respectively, the lag between exposure and infectiousness ($h_1 = 3$ days) and the delay in the build-up of the protective effect of vaccination ($h_2 = 5$ days). All epidemiological parameters are as reported in the captions of Figure 14 and Figure 15, which show the system evolution under the optimal control strategy.

The comparison between the two figures reveals the effect of increasing the penalty on disease burden by two orders of magnitude, from $w_i = 10^3$ to $w_i = 10^5$. In Figure 14, where the cost weight on infections is lower, the optimal control remains small throughout the horizon and the epidemic runs largely unimpeded: the infected compartments rise, peak, and decay on a timescale governed primarily by the intrinsic disease dynamics. The controller intervenes only mildly, consistent with the fact that, at $w_i = 10^3$, the marginal cost of an additional infected individual does not justify a large expenditure of control effort given that $w_u = 1$.

In Figure 15, by contrast, the higher weight $w_i = 10^5$ makes disease suppression far more valuable relative to intervention cost. The optimal control now activates much more aggressively and earlier, visibly altering the trajectories of the clinical compartments: the peaks of the infected compartments I, D, A, R, T are reduced compared to Figure 14.

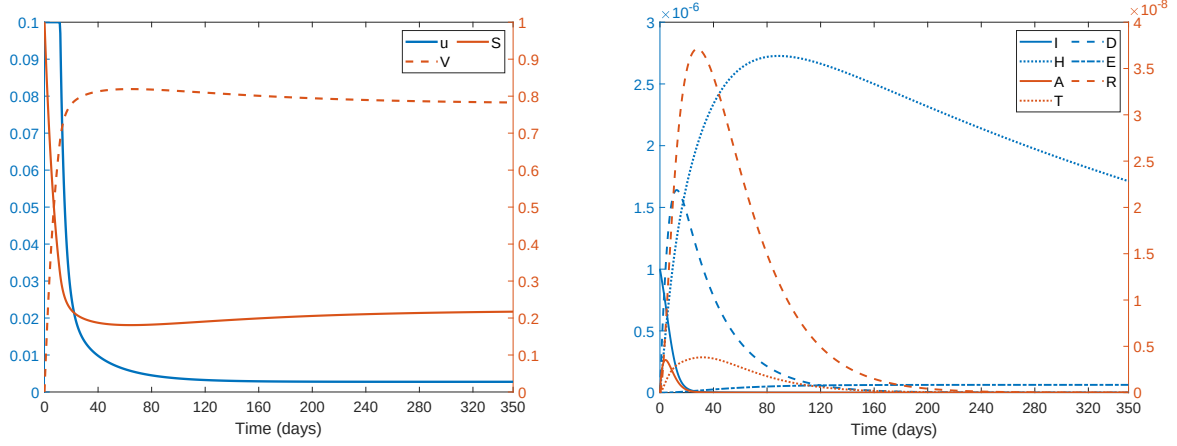


Figure 15: Extended SIDARTHE-V OCP with double delay, with the same parameters, control bounds and initial conditions as in Figure 14, and cost functional weights $w_i = 10^5$, $i = I, D, A, R, T$, and $w_u = 1$.

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