Structural Analysis of Dynamical Networks: BDC-Decomposition and Influence Matrix

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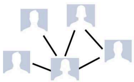
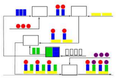
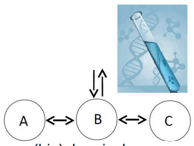
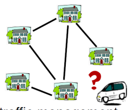

social networks

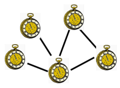

production and distribution systems

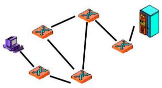


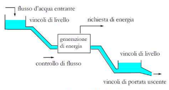
(bio)chemical processes

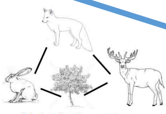

traffic management

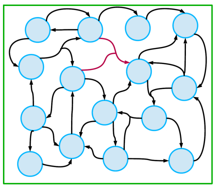
DYNAMICAL NETWORKS


synchronization


telecommunication / data communication networks


water distribution networks


biological systems and ecosystems


Local Interactions

Global Behaviour

Graph Representation

- Parameter-free approach: system structure

Structural analysis

Explain behaviours based on the system inherent structure (graph)

- Structurally assess fundamental properties

Structural properties

Satisfied by all the systems of a family specified by a structure, without numerical bounds.

- Applications to biochemical systems

Structural properties in nature

Biological systems → extremely robust: fundamental properties preserved despite huge uncertainties and parameter variations.

THE BDC-DECOMPOSITION

$\dot{x}(t) = Sg(x(t)) + g_0$, $\quad g$ monotonic functions

Local BDC-decomposition

The Jacobian can be decomposed as:

$$J(x) = \frac{\partial Sg(x)}{\partial x} = B\Delta(x)C, \qquad \Delta(x) = \text{diag}\left\{ \left| \frac{\partial g_k}{\partial x_h} \right| \right\} \succ 0.$$

The decomposition is unique (up to permutations).

Global BDC-decomposition

Given the equilibrium $\bar{x}$ ($0 = Sg(\bar{x}) + g_0$), $z \doteq x - \bar{x}$.
The system can be rewritten as:

$$\dot{z}(t) = Sg(z(t)+\bar{x}) - Sg(\bar{x}) = [BD(z)C]\; z(t), \qquad D(z) \succ 0.$$

(diagonal)

From local to global

For any vector $\bar{x}$

$$Sg(x) - Sg(\bar{x}) = \left[\int_0^1 J(\bar{x} + \sigma(x-\bar{x}))d\sigma\right](x-\bar{x}) = B\left[\int_0^1 D(\bar{x}+\sigma(x-\bar{x}))\; d\sigma\right] C\;(x-\bar{x})$$

Example: biochemical reaction network

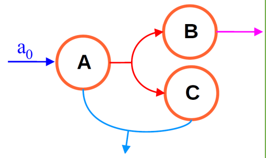
Species: A, B, C

Reactions: $\emptyset \xrightarrow{a_0} A$, $\; A \xrightarrow{g_a} B + C$, $\; A + C \xrightarrow{g_{ac}} \emptyset$, $\; B \xrightarrow{g_b} \emptyset$

Concentrations: a, b, c

ODE system:

$$\dot{a} = a_0 - g_a(a) - g_{ac}(a,c)$$
$$\dot{b} = g_a(a) - g_b(b)$$
$$\dot{c} = g_a(a) - g_{ac}(a,c)$$

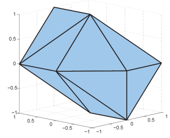


$$\begin{bmatrix}\dot{a}\\ \dot{b}\\ \dot{c}\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & -1 & 0\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}}_{S\,=\,\text{stoichiometric matrix}} \underbrace{\begin{bmatrix}g_a(a)\\ g_{ac}(a,c)\\ g_b(b)\end{bmatrix}}_{g\,=\,\text{rate}} + \underbrace{\begin{bmatrix}a_0\\ 0\\ 0\end{bmatrix}}_{g_0\,=\,\text{influx}}$$

$$\dot{z} = BDCz$$

$$D = \text{diag}\left\{\frac{\partial g_a}{\partial a}, \frac{\partial g_{ac}}{\partial a}, \frac{\partial g_{ac}}{\partial c}, \frac{\partial g_b}{\partial b}\right\} \succ 0$$

$$B = \begin{bmatrix}-1 & -1 & -1 & 0\\ 1 & 0 & 0 & -1\\ 1 & -1 & -1 & 0\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$



The procedure converges ⟹ structurally stable

$$\begin{bmatrix} + & 0 & - \\ + & + & - \\ ? & 0 & + \end{bmatrix}$$

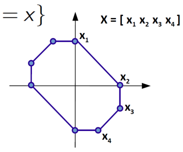
BDC-based computation of Polyhedral Lyapunov Functions for structural stability

Monotonic functions g and dissipative reactions $\frac{\partial \dot{x}_i}{\partial x_i} < 0$

exploit the BDC-decomposition! Structurally ⇔ for **any** $D_i > 0$

$$V_X(x) = \inf\{\|w\|_1 : \; Xw = x\}$$

X full row rank



Idea: $D(z(t)) \to D(t)$

Absorb the nonlinear system in a Linear Differential Inclusion

$$\dot{z}(t) = [BD(t)C]\; z(t), \qquad D(t) \succ 0. \qquad (LDI)$$

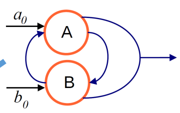
Any trajectory of the original system is also a trajectory of (LDI).

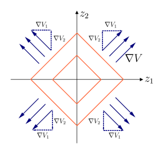
To analyse stability we can assume $0 \leq D_i \leq 1$.

Based on a discrete difference inclusion, a numerical algorithm computes the unit ball of the polyhedral Lyapunov function (if any) via set iteration.

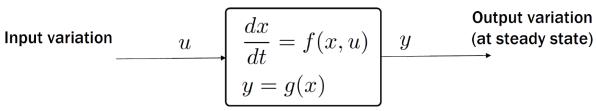
Polyhedral… why?

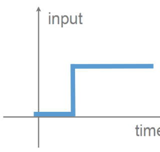
The only structural Lyapunov function is polyhedral!

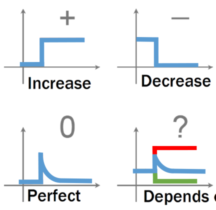




Structural Steady-State Influence: BDC-based computation







Structural Influence Matrix

Structural influence of variable j on variable i

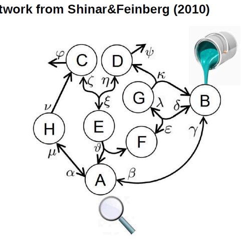
Assuming stability, $\Sigma_{ij} \in \{+, -, 0, ?\}$: sign of the steady-state variation of $x_i(\infty)$ due to a step input acting on x_j.

For systems admitting a BDC decomposition

$$\Sigma_{ij} = H_i(-BDC)^{-1}E_j,$$

H output matrix, E input matrix → efficient vertex algorithm

Network from Shinar&Feinberg (2010)



$$\begin{bmatrix} + & \boxed{+} & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & 0 & + & 0 & + & + \\ 0 & 0 & 0 & + & + & + & + & 0 \\ + & + & + & + & + & + & + & + \\ ? & ? & ? & ? & ? & ? & ? & ? \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \end{bmatrix}$$

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