Topology-Independent Robust Stability Conditions for Uncertain MIMO Networks

Carlos Andres Devia[a] and Giulia Giordano[b]

***Abstract*—We give a sufficient and a necessary condition for the topology-independent robust stability of networked systems formed by uncertain MIMO systems. Both conditions involve constants associated with the nominal node dynamics and arc interconnection matrices, the uncertainty bounds, and the maximum connectivity degree of the network; they are scalable (they can be checked locally), independent of the network topology and even of the number of nodes and arcs, and hold for networks of heterogeneous MIMO systems and interconnection matrices, with heterogeneous uncertainties. The dual cases of 1-norm and ∞-norm bounds are considered. In both cases, if the systems at the nodes are diagonal, we get a necessary and sufficient condition. We apply our results to the topology-independent robust stability analysis of a case-study from cancer biology.**

***Index Terms*—Network analysis and control, Stability of linear systems, Uncertain systems**

I. Introduction and Preliminaries

NETWORKS of dynamical systems, arising in multi-agent control [1], [2] and estimation [3], [4], and in the analysis of multi-compartment natural systems in biology, pharmacokinetics and epidemiology [5], [6], [7], [8], can be effectively analysed by studying the properties of the subsystems and of the interconnection graph. A widely studied problem is the stability of the whole dynamic network, given the stability of the subsystems, also in the presence of uncertainties.

In the frequency domain, robust stability conditions for interconnections of either SISO (single-input and single-output) or MIMO (multiple-input and multiple-output) linear systems were provided in [9], [10], [11], [12], [13] adopting Nyquist-type approaches and in [7], [8], [14], [15] using the generalised frequency variable framework; also, based on Integral Quadratic Constraints, [16], [17] provided scalable conditions that can be tested locally and used for control design [18]. Frequency-domain conditions for topology-independent robust stability were derived in [19], [20] for nominally homogeneous SISO systems and in [21] for homogeneous MIMO systems.

We consider the *state-space representation* of networked systems formed by uncertain MIMO systems interconnected through a directed graph with unknown topology. The network nodes are associated with heterogeneous MIMO systems, each with its own nominal state, input and output matrices belonging to a given set, and subject to bounded uncertainties. We give a sufficient and a necessary condition for topology-independent stability, robust with respect to the uncertainty in the dynamics and in the network topology, based on minimal information about the heterogeneous uncertain node systems:

- α: spectral abscissa (i.e., maximum real part of the eigenvalues) of the *nominal* state matrices;
- μ_B, μ_C: bounds on the *nominal* input and output matrices;
- ξ_A, ξ_B, ξ_C: *uncertainty bounds* for state, input and output matrices;

about the heterogeneous interconnection matrices:

- μ_G: bound on the *nominal* interconnection matrices;
- ξ_G: *uncertainty bound* for the interconnection matrices;

and about the network:

- $\mathscr{D}^*$: maximum connectivity degree;

the sufficient condition also requires the condition number χ^2 of some eigenmatrix of the nominal state matrices.

Then, all the networked systems in the family are stable if $\alpha+\chi^2\left[\xi_A+\mathscr{D}^*(\mu_B+\xi_B)(\mu_G+\xi_G)(\mu_C+\xi_C)\right]<0$ and only if $\alpha+\left[\xi_A+\mathscr{D}^*(\mu_B+\xi_B)(\mu_G+\xi_G)(\mu_C+\xi_C)\right]<0$.

Two dual scenarios are addressed: with 1-*norm* bounds, both conditions hold with $\mathscr{D}^*=\mathscr{D}^{\text{out}}$, the maximum *outward* connectivity degree; with ∞-*norm* bounds, they hold with $\mathscr{D}^*=\mathscr{D}^{\text{in}}$, the maximum *inward* connectivity degree.

In both cases, the gap between the sufficient and the necessary condition reveals the fundamental role of χ^2 in topology-independent stability. If the systems at the nodes are diagonal, then $\chi^2=1$ and the two inequalities become identical, yielding a necessary and sufficient condition for topology-independent robust stability. We also show how the minimum χ^2 can be computed, given the nominal state matrices at the nodes, to have the tightest possible gap.

The obtained conditions are conservative, but also scalable and easy to check locally: they apply even when the topology is unknown, independent of the number of nodes and arcs in the network, since they only rely on the maximum connectivity degree. Nodes and arcs can be added or removed in real time, in a plug-and-play fashion [22], [23], without compromising stability as long as the maximum connectivity degree remains the same. The nominal systems, the interconnection matrices and all the uncertainties can be heterogeneous, and even the number of states, inputs and outputs need not be the same, so the conditions are extremely general and can be applied to a very large class of networked systems.

Notation and Preliminaries. A directed graph with N nodes and M arcs is represented by the pair $\boldsymbol{\mathcal{G}}=\{\mathscr{N},\mathscr{A}\}$, where $\mathscr{N}=\{1,\dots,N\}$ is the node set and $\mathscr{A}\subset\mathscr{N}\times\mathscr{N}$ is the arc set, with $|\mathscr{A}|=M$, where $(i,j)\in\mathscr{A}$ denotes an arc that leaves node i and enters node j. Each node $i\in\mathscr{N}$ has an outward (resp. inward) connectivity degree δ_i^{out} (resp. δ_i^{in}), defined as the number of arcs that leave (resp. enter) the node. The maximum outward (resp. inward) connectivity degree is $\mathscr{D}^{\text{out}}=\max_{i\in\mathscr{N}}\delta_i^{\text{out}}$ (resp. $\mathscr{D}^{\text{in}}=\max_{i\in\mathscr{N}}\delta_i^{\text{in}}$).

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[a] Delft Center for Systems and Control, Delft University of Technology, The Netherlands. c.a.deviapinzon@tudelft.nl

[b] Department of Industrial Engineering, University of Trento, Italy. giulia.giordano@unitn.it

We denote by $\sigma(A)$ the spectrum of a square matrix A and by $\mathscr{K}_p(A) = \|A\|_p \|A^{-1}\|_p$ its condition number, where $\|A\|_p = \sup_{v \neq 0} \|Av\|_p / \|v\|_p$ denotes any matrix p-norm.

Theorem 1 (Bauer-Fike Theorem [24]). *Consider matrices A, $B \in \mathbb{R}^{n \times n}$, with A diagonalisable: $V^{-1}AV = diag\{\lambda_1, \dots, \lambda_n\}$ for some $V \in \mathbb{C}^{n \times n}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. For every (complex) eigenvalue ρ of $A + B$, there exists an index $k \in \{1, 2, \dots n\}$ such that $|\rho - \lambda_k| \leq \mathscr{K}_p(V)\|B\|_p$.*

Lemma 1. *Consider three matrices $\tilde{X}$, $\bar{X}$, $\delta X \in \mathbb{R}^{n \times n}$ such that $\tilde{X} = \bar{X} + \delta X$ and let $Z \in \mathbb{C}^{n \times n}$ be an eigenmatrix that diagonalises $\bar{X}$. Given the scalars $\varrho = \max_{\lambda \in \sigma(\bar{X})}\{Re(\lambda)\}$ and $\kappa \geq \mathscr{K}_p(Z)\|\delta X\|_p$, matrix $\tilde{X}$ is Hurwitz stable if*

$$\varrho + \kappa < 0. \tag{1}$$

Proof: Let $D(x, r)$ denote the closed disk with center $x \in \mathbb{C}$ and radius r. By Theorem 1, all the eigenvalues of $\tilde{X}$ are located in the set $\Upsilon = \bigcup_{\lambda \in \sigma(\bar{X})} D(\lambda, \kappa)$. Since $\max_{\varphi \in \Upsilon}\{\mathrm{Re}(\varphi)\} = \max_{\lambda \in \sigma(\bar{X})}\{\mathrm{Re}(\lambda)\} + \kappa = \varrho + \kappa$, all the eigenvalues of $\tilde{X}$ have negative real part if $\varrho + \kappa < 0$. ■

Since $\kappa \geq 0$, condition (1) requires $\varrho < 0$, i.e. Hurwitz stability of the nominal $\bar{X}$.

Denote by $\otimes$ the Kronecker product. We focus on the 1-norm, $\|X\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |X_{ij}|$, and the ∞-norm, $\|X\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |X_{ij}|$, of a matrix $X \in \mathbb{C}^{n \times m}$.

Theorem 2 (Properties of 1-norm and ∞-norm [25]). *Given complex matrices A and B of compatible dimensions, $\|AB\|_* \leq \|A\|_* \|B\|_*$ and $\|A \otimes B\|_* = \|A\|_* \|B\|_*$, where the subscript $*$ denotes either always 1 or always ∞.*

Lemma 2 (Norm of block-diagonal matrices). *The complex block-diagonal matrix $X = diag(X_k)_{k=1}^{K}$ has norm $\|X\|_* = \max_{k=1,\dots,K}\{\|X_k\|_*\}$, where the subscript $*$ denotes either always 1 or always ∞.*

II. Topology-Independent Robust Stability

We consider a family $\mathcal{N}$ of *uncertain networked systems*. The generic system in the family (cf. Figure 1) has an underlying graph structure $\mathcal{G} = \{\mathscr{N}, \mathscr{A}\}$, where each node in $\mathscr{N}$ is associated with an uncertain MIMO system and each arc in $\mathscr{A}$, labelled with an integer number in the set $\{1, \dots, M\}$, is associated with an uncertain interconnection matrix. Each node of $\mathcal{G}$ is associated with a linear MIMO system of the form

$$\dot{x}^{(i)} = A_i x^{(i)} + B_i u^{(i)}, \quad y^{(i)} = C_i x^{(i)}, \quad i \in \mathscr{N},$$

where the system matrices are the sum of a nominal and an uncertain part: $A_i = \bar{A}_i + \hat{A}_i$, $B_i = \bar{B}_i + \hat{B}_i$, $C_i = \bar{C}_i + \hat{C}_i$. The overall dynamics for the disconnected node systems is

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{2}$$

where $x = [{x^{(1)}}^\top \dots {x^{(N)}}^\top]^\top$, $u = [{u^{(1)}}^\top \dots {u^{(N)}}^\top]^\top$, $y = [{y^{(1)}}^\top \dots {y^{(N)}}^\top]^\top$, $A = \mathrm{diag}(A_i)_{i \in \mathscr{N}}$, $B = \mathrm{diag}(B_i)_{i \in \mathscr{N}}$, $C = \mathrm{diag}(C_i)_{i \in \mathscr{N}}$. Splitting nominal and uncertain parts, $A = \bar{A} + \hat{A}$, where $\bar{A} = \mathrm{diag}(\bar{A}_i)_{i \in \mathscr{N}}$, $\hat{A} = \mathrm{diag}(\hat{A}_i)_{i \in \mathscr{N}}$; $B = \bar{B} + \hat{B}$, where $\bar{B} = \mathrm{diag}(\bar{B}_i)_{i \in \mathscr{N}}$, $\hat{B} = \mathrm{diag}(\hat{B}_i)_{i \in \mathscr{N}}$; $C = \bar{C} + \hat{C}$, where $\bar{C} = \mathrm{diag}(\bar{C}_i)_{i \in \mathscr{N}}$, $\hat{C} = \mathrm{diag}(\hat{C}_i)_{i \in \mathscr{N}}$.

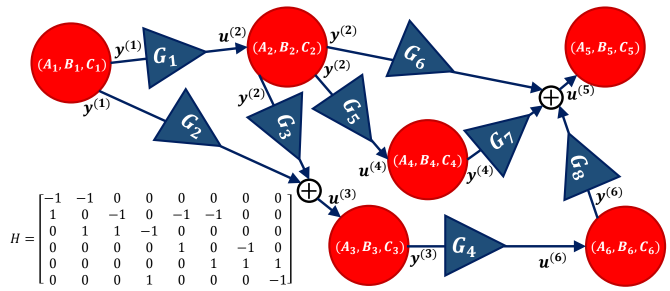

Fig. 1: Example of an uncertain networked system in the family $\mathcal{N}$, with $N = 6$ node systems, $M = 8$ arcs and incidence matrix H.

The node systems are connected through the incidence matrix $H \in \{-1, 0, 1\}^{N \times M}$ of $\mathcal{G}$, defined as $H_{ih} = 1$ if the arc $h \in \mathscr{A}$ *enters* node $i \in \mathscr{N}$; $H_{ih} = -1$ if the arc h *leaves* node i; and $H_{ih} = 0$ otherwise. In particular, we define matrices $P = \max\{H, 0\}$ and $R = -\min\{H, 0\}$ elementwise, so that $P_{ih} = 1$ if arc h enters node i and $R_{ih} = 1$ if arc h leaves node i. These scalar matrix entries match nodes and arcs according to the interconnection topology. We denote by G_h the interconnection matrix, of the proper size, associated with arc h, which is the sum of a nominal and an uncertain part: $G_h = \bar{G}_h + \hat{G}_h$. Then, the input to node $i \in \mathscr{N}$ is

$$u^{(i)} = \sum_{h=1}^{M} P_{ih} G_h \Big(\sum_{j=1}^{N} R_{jh} y^{(j)} \Big),$$

where just one of the scalars R_{jh}, with $j = 1, \dots, N$, is nonzero and selects the node output "feeding" arc h.

Let $G = \mathrm{diag}(G_h)_{h \in \mathscr{A}}$. Assume that all the nodes have the same number of inputs and outputs (this simplifies the notation, but is not necessary for the results to hold, as discussed in Section III). Then, compactly,

$$u = (P \otimes I_p) G (R^\top \otimes I_q) y, \tag{3}$$

where p is the number of inputs and q is the number of outputs of each node.

Merging (2) and (3) gives the networked system

$$\dot{x} = \big[A + B(P \otimes I_p) G (R^\top \otimes I_q) C\big] x \doteq \tilde{A} x. \tag{4}$$

The elements of the networked system family $\mathcal{N}$ can be associated with different graphs, having a different number of nodes and arcs, as long as the connectivity degree is bounded.

Assumption 1. *For each system in the family $\mathcal{N}$, the maximum outward (resp. inward) connectivity degree of the underlying graph is at most $\mathscr{D}^{out}$ (resp. $\mathscr{D}^{in}$).*

Assumption 2. *For each system in the family $\mathcal{N}$, all node systems (A_i, B_i, C_i), for $i \in \mathscr{N}$, are such that $A_i = \bar{A}_i + \hat{A}_i$, $B_i = \bar{B}_i + \hat{B}_i$, $C_i = \bar{C}_i + \hat{C}_i$, where, denoting with $*$ either always 1 or always ∞,*
- *$\max_{\lambda \in \sigma(\bar{A}_i)}\{Re(\lambda)\} \leq \alpha$, for a given $\alpha < 0$;*
- *$\|W_i\|_* \leq \chi$ and $\|W_i^{-1}\|_* \leq \chi$, for a given $\chi \geq 1$, where W_i is some eigenmatrix that diagonalises $\bar{A}_i$;*
- *$\|\bar{B}_i\|_* \leq \mu_B$ and $\|\bar{C}_i\|_* \leq \mu_C$, for given μ_B, $\mu_C > 0$;*
- *$\|\hat{A}_i\|_* \leq \xi_A$, $\|\hat{B}_i\|_* \leq \xi_B$ and $\|\hat{C}_i\|_* \leq \xi_C$, for given ξ_A, ξ_B, $\xi_C \geq 0$.*

Assumption 2 implies $\mathscr{K}_*(W_i) = \|W_i\|_* \|W_i^{-1}\|_* \leq \chi^2$.

Assumption 3. *For each system in the family $\mathcal{N}$, all interconnection matrices G_h, for $h \in \mathscr{A}$, are such that $G_h = \bar{G}_h + \hat{G}_h$, with $\|\bar{G}_h\|_* \leq \mu_G$ and $\|\hat{G}_h\|_* \leq \xi_G$, for given μ_G, $\xi_G \geq 0$, where the subscript $*$ denotes either always 1 or always ∞.*

We are then ready to state our main results for the case of uncertain networked systems with 1-norm bounds; the proofs are given in Section III.

Theorem 3 (Sufficient condition for topology-independent robust stability). *Consider the family of networked systems $\mathcal{N}$, under Assumptions 1, 2 and 3 with 1-norm bounds. Then, all systems in $\mathcal{N}$ are stable if*

$$\alpha + \chi^2 \left[\xi_A + \mathscr{D}^{out}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right] < 0. \quad (5)$$

Theorem 4 (Necessary condition for topology-independent robust stability). *Consider the family of networked systems $\mathcal{N}$, under Assumptions 1, 2 and 3 with 1-norm bounds. A necessary condition for all systems in $\mathcal{N}$ to be stable is*

$$\alpha + \left[\xi_A + \mathscr{D}^{out}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right] < 0. \quad (6)$$

By duality, our main results still hold if the 1-norm is replaced by the ∞-norm, and $\mathscr{D}^{\text{out}}$ is replaced by $\mathscr{D}^{\text{in}}$. Since the proofs are essentially unchanged, we just report the results.

Proposition 1 (Dual of Theorem 3). *Consider the family of networked systems $\mathcal{N}$, under Assumptions 1, 2 and 3 with ∞-norm bounds. Then, all systems in $\mathcal{N}$ are stable if*

$$\alpha + \chi^2 \left[\xi_A + \mathscr{D}^{in}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right] < 0. \quad (7)$$

Proposition 2 (Dual of Theorem 4). *Consider the family of networked systems $\mathcal{N}$, under Assumptions 1, 2 and 3 with ∞-norm bounds. Then, a necessary condition for all systems in $\mathcal{N}$ to be stable is that*

$$\alpha + \left[\xi_A + \mathscr{D}^{in}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right] < 0. \quad (8)$$

For diagonal systems, the topology-independent robust stability condition becomes necessary and sufficient.

Proposition 3 (Diagonal systems). *Consider the family of networked systems $\mathcal{N}$, under Assumptions 1, 2 and 3 with 1-norm (resp. ∞-norm) bounds. Assume that, for each element of the family, all the systems at the nodes have a diagonal state matrix A_i. Then, all systems in $\mathcal{N}$ are stable if and only if inequality* (5) *(resp.* (7)*) holds.*

Proof: For diagonal systems, $\chi^2 = 1$. Then, the result directly follows from Theorems 3 and 4 in the 1-norm case, and from Propositions 1 and 2 in the ∞-norm case. ∎

Our results highlight the crucial role of the condition number χ^2 for topology-independent stability: for non-diagonal systems, it leads to a gap between the sufficient and the necessary condition, thus introducing conservativeness. To have the tightest gap, we wish to compute the *minimum* value of χ^2. Consider the nominal matrix $\bar{A}$ corresponding to the system associated with a *single node*; being diagonalisable, it has distinct eigenvectors. Then, the columns of its eigenmatrix W can be scaled independently with the positive diagonal matrix $D = \text{diag}(D_i)$ and we can find

$$(\chi^2)^{\text{opt}} = \min_{D \in \text{diag}(D_i), D_i > 0} \|WD\|_* \|D^{-1}W^{-1}\|_* \quad (9)$$

where the subscript $*$ denotes either always 1 or always ∞.

This optimisation problem has a neat solution.

Proposition 4 (Minimum χ^2). *The optimal $(\chi^2)^{opt}$ in* (9) *is obtained when D is such that: all columns of $\tilde{W} = DW$ have unitary 1-norm, with 1-norm bounds; all rows of $\tilde{W}^{-1} = D^{-1}W^{-1}$ have unitary 1-norm, with ∞-norm bounds.*

Proof: Set $U = W^{-1}$ and denote by W_j the jth column of W, by U_i the ith row of U. Then, $\|WD\|_1 \|D^{-1}U\|_1 = \max_j D_j \|W_j\|_1 \max_h \sum_i \frac{|U_{ih}|}{D_i} = \max_j \frac{1}{z_j} \max_h \nu_h^\top z$, where the last equality follows by assuming without restriction that $\|W_j\|_1 = 1$ (which can be obtained via pre-scaling) and denoting by z the vector with ith component $z_i = 1/D_i$ and by ν_h the vector with ith component $|U_{ih}|$. In the dual case, assuming $\|U_h\|_1 = 1$ without restriction, $\|WD\|_\infty \|D^{-1}U\|_\infty = \max_i \sum_j |W_{ij}| D_j \max_h \frac{\|U_h\|_1}{D_h} = \max_i \nu_i^\top z \max_h \frac{1}{z_h}$, where vector z has ith component $z_i = D_i$ and vector ν_i has jth component $|W_{ij}|$.

Then the function to be minimised can be written in the form $\phi(z) = \max_j \left\{\frac{1}{z_j}\right\} \max_h \left\{\nu_h^\top z\right\}$, where ν_h are non-negative vectors and $z > 0$ componentwise. Since ϕ is positively homogeneous of order 0 (i.e., $\phi(\lambda z) = \phi(z)$ for any $\lambda > 0$), we can find its minimum assuming the additional constraint

$$\max_j \left\{\frac{1}{z_j}\right\} = 1. \quad (10)$$

Indeed, if $z^{\text{opt}} > 0$ is a minimum, then we can take the maximum $1/z_{i^*}^{\text{opt}} = \max_j 1/z_j^{\text{opt}}$ and set $\lambda \doteq 1/z_{i^*}^{\text{opt}} \geq 1/z_j^{\text{opt}}$ for all j. Now, λz^{opt} produces the same minimum value, since $\phi(\lambda z^{\text{opt}}) = \phi(z^{\text{opt}})$, and satisfies (10). Therefore, the additional constraint does not change the result. The surface in (10) can be split into n faces: $\mathscr{F}_i = \{z \colon z_i = 1,\ z_j \geq 1,\ j \neq i\}$, for $i = 1, \ldots, n$. So we need to consider n problems of the form $\min_z \max_h \left\{\nu_h^\top z\right\}$ with constraints $z_i = 1$ for $i = 1, \ldots, n$ and $z_j \geq 1$ for $j = 1, \ldots, n$, $j \neq i$, which can be converted into linear programs. Since all the components of ν_h are non-negative, the minimum of the ith problem with $z_i = 1$ is immediately achieved by choosing the smallest possible value for all other components: $z_j = 1$ for all $j \neq i$. Hence the initial pre-scaling, with $\|W_j\|_1 = 1$ in the 1-norm case and $\|U_h\|_1 = 1$ in the ∞-norm case, was already optimal. ∎

III. Proofs of the Main Results

Proof of Theorem 3: We assume for simplicity that all the nodes have the same number of inputs, p, and outputs, q; the general case is discussed in Section III. To assess stability of the networked system (4), we rewrite matrix $\tilde{A}$ as the sum of three matrices: $\tilde{A} = \bar{A} + \delta A_1 + \delta A_2$, where $\delta A_1 = \hat{A}$ represents the uncertainty in the state dynamics and $\delta A_2 = B(P \otimes I_p)G(R^\top \otimes I_q)C$ includes the uncertainty due to the input and output matrices and to the interconnection. Thanks to its particular block-diagonal structure, the nominal matrix $\bar{A}$ can be diagonalised as $\bar{A} = W^{-1}\Lambda W$, where $\Lambda = \text{diag}(\Lambda_i)_{i \in \mathcal{N}}$ has on the diagonal the blocks $\Lambda_i = \text{diag}(\lambda)_{\lambda \in \sigma(\bar{A}_i)}$ including the eigenvalues of the individual systems at the nodes, while $W = \text{diag}(W_i)_{i \in \mathcal{N}}$ has on the diagonal the eigenmatrices W_i of $\bar{A}_i$ that satisfy $\|W_i\|_1 \leq \chi$ and $\|W_i^{-1}\|_1 \leq \chi$ as per Assumption 2 with 1-norm bounds.

The stability of $\tilde{A}$ can be checked by applying Lemma 1 with $\tilde{X} = \tilde{A}$, $\bar{X} = \bar{A}$, $\delta X = \delta A_1 + \delta A_2$, $Z = W$, $\varrho = \alpha$, and $\kappa = \chi^2 \left[\xi_A + \mathscr{D}^{\text{out}}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right]$. In fact, since $\max_{\lambda \in \sigma(\bar{A})}\{\text{Re}(\lambda)\} \leq \alpha$, which is negative in view of Assumption 2, the nominal state matrices are Hurwitz stable. To make sure that the assumptions of Lemma 1 are all satisfied, we must show that

$$\mathscr{K}_1(W)\|\delta A_1 + \delta A_2\|_1 \leq \kappa, \tag{11}$$

with $\kappa = \chi^2 \left[\xi_A + \mathscr{D}^{\text{out}}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\right]$. We have $\mathscr{K}_1(W) \leq \chi^2$ in view of Lemma 2, while $\|\delta A_1 + \delta A_2\|_1 \leq \|\delta A_1\|_1 + \|\delta A_2\|_1$ can be upper bounded by exploiting Theorem 2 and, in view of the block structure of matrix δA_1, Lemma 2: $\|\delta A_1\|_1 = \|\hat{A}\|_1 = \max_{i \in \mathcal{N}}\{\|\hat{A}_i\|_1\} \leq \xi_A$ and $\|\delta A_2\|_1 \leq \|(P \otimes I_p)\|_1 \|(R^\top \otimes I_q)\|_1 \|B\|_1 \|G\|_1 \|C\|_1 \leq \mathscr{D}^{\text{out}}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)$, where the last inequality holds because

$$\|(P \otimes I_p)\|_1 \|(R^\top \otimes I_q)\|_1 = \|P\|_1 \|R^\top\|_1 \leq \mathscr{D}^{\text{out}} \tag{12}$$

and $\|B\|_1 = \|\bar{B} + \hat{B}\|_1 \leq \|\bar{B}\|_1 + \|\hat{B}\|_1 = \max_{i \in \mathcal{N}}\{\|\bar{B}_i\|_1\} + \max_{i \in \mathcal{N}}\{\|\hat{B}_i\|_1\} \leq \mu_B + \xi_B$, $\|G\|_1 = \|\bar{G} + \hat{G}\|_1 \leq \|\bar{G}\|_1 + \|\hat{G}\|_1 = \max_{h \in \mathscr{A}}\{\|\bar{G}_h\|_1\} + \max_{h \in \mathscr{A}}\{\|\hat{G}_h\|_1\} \leq \mu_G + \xi_G$, $\|C\|_1 = \|\bar{C} + \hat{C}\|_1 \leq \|\bar{C}\|_1 + \|\hat{C}\|_1 = \max_{i \in \mathcal{N}}\{\|\bar{C}_i\|_1\} + \max_{i \in \mathcal{N}}\{\|\hat{C}_i\|_1\} \leq \mu_C + \xi_C$. Since inequality (11) is proven, Lemma 1 can be applied and guarantees that matrix $\tilde{A}$ in (4) is Hurwitz stable if the sufficient condition (5) is satisfied. □

A remark on generality: All our results hold unchanged if, for the generic networked system in the family $\boldsymbol{\mathcal{N}}$ with N nodes, the number of inputs and outputs of the nodes are $(p_1, \ldots, p_N)$ and $(q_1, \ldots, q_N)$ respectively. Then, the expression in (3) can be replaced by $u = \mathscr{P} G \mathscr{R} y$ and the networked system in (4) by $\dot{x} = [A + B\mathscr{P} G \mathscr{R} C]x$, where the matrices $\mathscr{P}$ and $\mathscr{R}$ are built as follows. The block matrix in the position (i, j) of $\mathscr{P}$ is the square scaled identity matrix $P_{ij} I_{p_i}$ if $P_{ij} = 1$, while if $P_{ij} = 0$ it is a rectangular matrix of zeros of the appropriate size. For $\mathscr{R}$, the block matrix in position (i, j) is $R_{ji} I_{q_j}$ if $R_{ji} = 1$, while it is a rectangular zero matrix if $R_{ji} = 0$.

Example 1. *Consider a network composed of* 3 *nodes, with* $p_1 = 2$, $p_2 = 3$, $p_3 = 2$ *inputs and* $q_1 = 1$, $q_2 = 2$, $q_3 = 1$ *outputs. Let the incidence matrix be*

$$H = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

that is, the network has 4 *arcs with matrices* $G_1 \in \mathbb{R}^{3 \times 1}$, $G_2 \in \mathbb{R}^{2 \times 2}$, $G_3 \in \mathbb{R}^{2 \times 1}$, *and* $G_4 \in \mathbb{R}^{2 \times 2}$. *Then*

$$P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathscr{P} = \left[\begin{array}{ccc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since $\|\mathscr{P}\|_1 = 1$ and $\|\mathscr{R}\|_1 \leq \mathscr{D}^{\text{out}}$, replacing $(P \otimes I_p)$ with $\mathscr{P}$ and $(R^\top \otimes I_q)$ with $\mathscr{R}$ in (12) does not affect the result.

Therefore, Theorem 3 holds even when the nodes have different number of states, inputs and outputs. □

Proof of Theorem 4: If condition (6) is violated, hence

$$\alpha + \big(\xi_A + \mathscr{D}^{\text{out}}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)\big) \geq 0, \tag{13}$$

then there exists an unstable system structure in the family $\boldsymbol{\mathcal{N}}$. We show that this structure is associated with a circulant matrix, for which the following result [26, Sec. 3.1] holds.

Theorem 5 (Spectrum of a circulant matrix.). *The eigenvalues of a circulant matrix* $C \in \mathbb{R}^{n \times n}$ *with coefficients* $\{c_0, c_1, \ldots, c_{n-1}\}$,

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix},$$

are

$$\psi_m = \sum_{k=0}^{n-1} c_k \rho_m^k, \tag{14}$$

where $\rho_m = exp(\frac{-2\pi i m}{n})$, $m \in \{0, \ldots, n-1\}$.

Without loss of generality, since the networked systems in the family $\boldsymbol{\mathcal{N}}$ can have node systems of *any size*, we consider a networked system where each node is the same scalar system $\dot{x}^{(i)} = a x^{(i)} + b u^{(i)}$, $y^{(i)} = c x^{(i)}$, $i \in \mathcal{N}$, and all the arcs are associated with the same interconnection scalar g. Assume that $\mathscr{D}^{\text{out}}$ arcs leave each node to reach the previous $\mathscr{D}^{\text{out}}$ nodes: there is an arc leaving node i to node $(i - k + N) \mod N$ (with the understanding that node 0 corresponds to node N) for $i \in \mathcal{N}$ and $k = 1, \ldots, \mathscr{D}^{\text{out}}$.

The networked system has the following state matrix: $\tilde{A} = aI_N + (bI_N)(P)(gI_M)(R^\top)(cI_N) = aI_N + bgcPR^\top$, where $[PR^\top]_{ij} = 1$ if there is an arc going from node j to i, $[PR^\top]_{ij} = 0$ otherwise. For this graph structure, $\tilde{A}$ is a circulant matrix satisfying Theorem 5, where $c_0 = a$, $c_k = bgc$ for $k = 1, \ldots, \mathscr{D}^{\text{out}}$ and the other coefficients are zero. Then, equation (14) with $m = 0$ gives $\psi_0 = \sum_{k=0}^{n-1} c_k = a + \mathscr{D}^{\text{out}} bgc = \alpha + \xi_A + \mathscr{D}^{\text{out}}(\mu_B + \xi_B)(\mu_G + \xi_G)(\mu_C + \xi_C)$, where the last equality is obtained by splitting the *nominal* and the *uncertain* part in $a = \alpha + \xi_A$, $b = \mu_B + \xi_B$, $c = \mu_C + \xi_C$, and $g = \mu_G + \xi_G$. By inequality (13) the eigenvalue ψ_0 of matrix $\tilde{A}$ is nonnegative, therefore the networked system is unstable. Since this system belongs to the family $\boldsymbol{\mathcal{N}}$, this proves the necessity of condition (6). □

IV. An Application to Cancer Biology

Consider a multi-compartment evolutionary model describing growth, mutation and metastasis of a heterogeneous tumor cell population [27], where a set of mutant cell lines $\mathscr{M}$ can spread in a set of body compartments $\mathscr{J}$ and the d available drugs are differently effective against different mutants in different compartments. The mutants can settle just in some compartments: $\mathscr{M}_k$ denotes the set of mutants in compartment k. Denoting by r_i^k the growth rate of mutant i in compartment k, q_{ji}^k the mutation rate from mutant j to i in compartment k, μ_i^{ck} the migration rate from compartment c to k of mutant i ($\mu_i^{ck} = 0$ if there is no migration path), $\phi_{s,i}^k$ the effect of drug s on mutant i in compartment k, and ℓ_s the constant amount of drug s, the concentration x_i^k of mutant $i \in \mathscr{M}_k$ in compartment

$k \in \mathcal{J}$ evolves as $\dot{x}_i^k = \sum_{j\in\mathcal{M}_k} r_i^k q_{ji}^k x_j^k + \sum_{c\in\mathcal{J}} r_i^k \mu_i^{ck} x_i^c - \sum_{j\in\mathcal{M}_k} q_{ij}^k x_i^k - \sum_{c\in\mathcal{J}} \mu_i^{kc} x_i^k - \sum_{s=1}^{d} \phi_{s,i}^k \ell_s x_i^k$.

We can see this model as a networked system with compartments (nodes), including a set of mutants, connected by possible migration routes (arcs). Compartment k is associated with the linear system $\dot{x}^k = A_k x^k + B_k u^k$, $y^k = x^k$, where $x^k = (x_i^k)_{i\in\mathcal{M}_k}$ includes all mutant lines in compartment k and $u^k = (u_i^k)_{i\in\mathcal{M}_k}$, where u_i^k is the sum of all cells of mutant i migrating to compartment k. For the state matrix, $[A_k]_{ii} = r_g^k q_{gg}^k - \sum_{j\in\mathcal{M}_k, g\neq j} q_{gj}^k - \sum_{c\in\mathcal{J}, c\neq k} \mu_g^{kc} - \sum_{s=1}^{d} \phi_{s,g}^k \ell_s$, with $g = \mathcal{M}_k(i)$, while $[A_k]_{ij} = r_g^k q_{fg}^k$, with $g = \mathcal{M}_k(i)$, $f = \mathcal{M}_k(j)$. The nonzero entries of B_k are $[B_k]_{ii} = r_g^k$, with $g = \mathcal{M}_k(i)$. The nonzero entries of the interconnection matrix G_h, associated with the arc from compartment k to compartment c, are $[G_h]_{ij} = \mu_g^{kc}$ if $g = \mathcal{M}_c(i) = \mathcal{M}_k(j)$.

As in all biological systems, the parameter values are subject to huge uncertainties. The network topology, and even the number of affected compartments, are not known exactly. However, if we assume that the overall networked system belongs to the family $\mathcal{N}$ satisfying Assumptions 1, 2 and 3 with 1-norm bounds, with $\mathcal{D}^{\text{out}} = 3$ (mutants in a compartment can migrate to 3 other compartments at most), $\alpha = -25.0227$, $\chi = 1.2236$, $\mu_B = 6.2$, $\mu_C = 1$, $\mu_G = 0.3$, $\xi_A = 8.5268$, $\xi_B = 0.93$, $\xi_C = 0$, $\xi_G = 0.045$, then condition (5) is satisfied: $\alpha + \chi^2\left[\xi_A + \mathcal{D}^{\text{out}}(\mu_B+\xi_B)(\mu_G+\xi_G)(\mu_C+\xi_C)\right] = -1.2 < 0$. As long as the networked system belongs to this class, *stability is robustly guaranteed* (namely, the adopted cancer therapy successfully reduces the tumor size) *for all topologies* with maximum degree 3, *regardless of the number of nodes* (affected body compartments) *and arcs* (possible migration paths), *and even of the actual number of inputs, outputs and states for each node* (number of mutants in each compartment).

For comparative simulations, we consider $\mathcal{J} = \{1,2,3,4\}$ and $\mathcal{M} = \{1,2,3\}$ with $\mathcal{M}_1 = \{1,2\}$, $\mathcal{M}_2 = \{2,3\}$, $\mathcal{M}_3 = \{1,2,3\}$, $\mathcal{M}_4 = \{1,3\}$. We take the uncertain parameters in the same intervals for all compartments: $r_1q_{11} \in [2.4, 3.3]$, $r_2q_{12} \in [0.61, 0.82]$, $r_3q_{13} \in [0.76, 1]$, $r_1q_{21} \in [0.24, 0.33]$, $r_2q_{22} \in [3.1, 4.1]$, $r_3q_{23} \in [1.1, 1.5]$, $r_1q_{31} \in [0.73, 0.98]$, $r_2q_{32} \in [0.31, 0.41]$, $r_3q_{33} \in [3.8, 5.1]$, $q_{11}, q_{22}, q_{33} \in [0.65, 0.78]$, $q_{12}, q_{13} \in [0.13, 0.16]$, $q_{21}, q_{32} \in [0.065, 0.078]$, $q_{23}, q_{31} \in [0.2, 0.23]$, $r_1 \in [2.8, 5.2]$, $r_2 \in [3.5, 6.5]$, $r_3 \in [4.3, 8.1]$, $\mu_1 \in [0.19, 0.21]$, $\mu_2 \in [0.29, 0.31]$, $\mu_3 \in [0.099, 0.1]$, $\phi_{1,1} \in [0.3981, 0.4019]$, $\phi_{2,1} \in [0.497, 0.503]$, $\phi_{1,2} \in [0.1592, 0.1608]$, $\phi_{2,2} \in [0.1988, 0.2012]$, $\phi_{1,3} \in [0.199, 0.201]$, $\phi_{2,3} \in [0.2485, 0.2515]$. Figure 2 shows the graph representation of the system with all the possible 12 mutation paths: each mutation path can be active or inactive (hence $\mu_i = 0$), leading to 4096 different graph topologies.

With 2 available drugs, we compare four different therapies: $T_1 = \{\ell_1 = 1.957, \ell_2 = 21.137\}$, $T_2 = \{\ell_1 = 2.571, \ell_2 = 26.453\}$, $T_3 = \{\ell_1 = 3.531, \ell_2 = 29.302\}$, $T_4 = \{\ell_1 = 11.76, \ell_2 = 133.229\}$. Only with therapy T_4 the uncertain networked system satisfies the sufficient condition (5).

As shown in Table I, T_1 stabilises the nominal disconnected systems, but can fail in the presence of uncertainties and/or interconnections; T_2 guarantees robust stability of the disconnected systems, but can fail when the systems are interconnected; T_3 guarantees stability also of all the interconnected systems, but not robustly; finally, T_4 guarantees topology-independent robust stability, as expected. Figure 3, showing the time evolution of the total number of cancer cells, and Figure 4, showing the eigenvalue distribution, confirm that only therapy T_4 ensures stability for all network topologies and all uncertainty realisations; for all other therapies, at least

	T_1	T_2	T_3	T_4
Case 1: disconnected, nominal	**S**	**S**	**S**	**S**
Case 2: disconnected, uncertain	**U**	**S**	**S**	**S**
Case 3: connected, nominal	**U**	**U**	**S**	**S**
Case 4: connected, uncertain	**U**	**U**	**U**	**S**

TABLE I: Effect of the therapies in different cases. **S**: stability is guaranteed for all systems in the case. **U**: at least one system in the case is unstable. Simulations with 150 random parameter variations for uncertain disconnected systems and with all the 4096 possible interconnection topologies for connected systems (each with 2 parameter variations in the uncertain case).

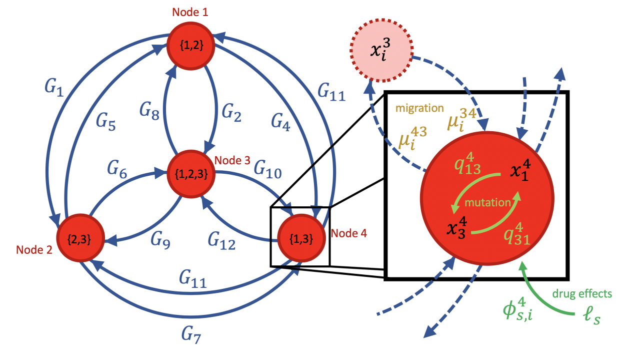

Fig. 2: Example of possible graph topology for multi-compartment cancer evolution, with 4 nodes (body compartments) and 12 directed arcs (migration paths); illustration of the mutation, migration and drug selection dynamics.

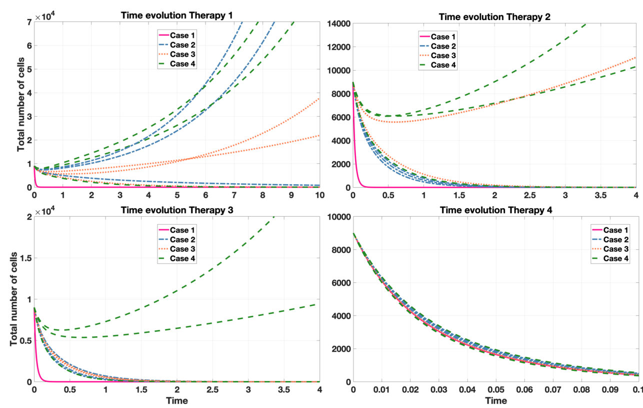

Fig. 3: Time evolution for the system cases and therapies in Table I.

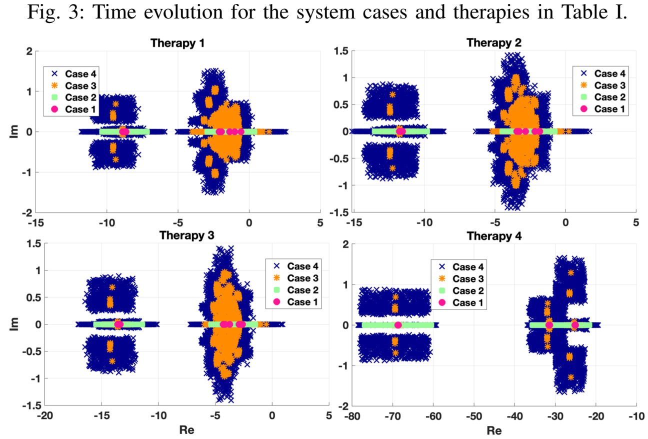

Fig. 4: Eigenvalue distribution for the system cases and therapies in Table I.

one system realisation is unstable, meaning that the chosen treatment fails and the tumor grows unbounded.

V. Discussion and Outlook

We deal with the topology-independent robust stability analysis of uncertain networked systems with completely unknown topology (but known maximum connectivity degree). Both the necessary and the sufficient condition provided here are easy to verify in a state-space framework and are fully scalable, since they can be checked locally and do not depend on the number of nodes and arcs. Both the systems at the nodes and their uncertainties, as well as the uncertain interconnection matrices at the arcs, can be heterogeneous, thus making these conditions applicable to a general class of systems.

However, our results are conservative, because they cannot exploit the knowledge of the physical structure of the uncertainties in the system parameters and interconnections, and tight norm bounds on the system matrices are hard to obtain.

Since we seek *topology-independent* results, another unavoidable source of conservativeness is that we cannot exploit the knowledge of the interconnection and its possibly stabilising effects. Hence, requiring stability of the individual subsystems is necessary for topology-independent stability: the system with disconnected nodes is a possible topology. The interconnection may compromise the stability of the node systems (as shown in the case-study in Section IV). The maximum connectivity degree naturally appears in our conditions; intuitively, a smaller degree facilitates topology-independent stability, because it limits the number of possible topologies among which the worst case must be considered.

For given nominal systems and uncertainty bounds, conditions (5) or (7) allow to find the maximum connectivity degree ensuring topology-independent robust stability; the stability of the networked system is robust to online modifications of the network, in a plug-and-play framework [22], [23], provided that the maximum connectivity degree is not exceeded.

Inequalities (5) and (7) can then be seen as a balance between the stable systems at the nodes, on the one hand, and the uncertainties and interconnections that can potentially destabilise the overall system, on the other hand. The sufficient condition may not be satisfied because the spectral abscissa α of the nominal systems at the nodes is not negative enough to counteract the possibly destabilising effect of interconnections and uncertainties. Then, local controllers can be added to move the eigenvalues further to the left of the complex plane, until the sufficient condition is met. An interesting future direction is to find the optimal local controller that minimises the left-hand-side of (5) and (7) with respect to the spectral abscissa α and the condition number χ^2.

Linear systems have been considered in this paper. Future work includes the extension to special classes of nonlinear systems, such as input-affine systems and Lur'e systems. Also, it will be interesting to merge the proposed robustness analysis with network objectives, such as consensus or synchronisation.

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