A robust decentralized control for channel sharing communication

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***Abstract*—We consider the problem of controlling the transmission rate in a communication network where each node adjusts its own transmission rate exclusively based on the physical medium occupation (the band occupancy of the aggregate complementary nodes). We show how to design a decentralized control for maximizing both band occupancy and fairness. If the network is fully connected, the problem admits a simple solution. Difficulties arise in the case of partially connected networks, and in the presence of time–varying network topologies and delays. General conditions are given which, by properly tuning control parameters, assure stability. These conditions are conservative and affect the system performance. However, we show that, in the case of symmetric connections, stability can be studied based on the system eigenvalues even in the presence of topology switchings. Less conservative bounds can be inferred by exploiting known properties of the eigenvalues of the adjacency matrix of a graph. We finally consider the multi–channel case, in which nodes may jump among channels: the previous scheme can be extended to this case and asymptotically ensures uniform channel exploitation.** [1]

Keywords: Distributed control, Transmission control, Switching, Lyapunov Functions.

I. Motivations and description of the control problem

In highly dynamic mobile scenarios (*e.g.* vehicle–to–vehicle communication [2], swarm robot systems [3], and affinity matching [4], [5]) the communication protocol should be able to adapt to sudden changes in network topology. Moreover, often no fraction of the overall available band can be used for network discovery, coordination and connection setup, hence the control must be decentralized. Furthermore, the control should possibly be simple, in order to be implemented on low-computation-power devices with limited energy consumption. In addition, simple control algorithms do not suffer of fragility phenomena in the presence of network variations [6]. The Carrier Sense Multiple Access (CSMA) p-persistent protocol family represents an interesting solution for coordinator–free *ad hoc* networks [7], [8], [9], [2], [10], [11]. In wireless communication networks, fairness is not less important than the maximization of channel utilization [12], [13], [14]; concurrently fulfilling both design goals is challenging and crucial.

In a recent paper [15] an improvement of the p-persistent protocol has been proposed based on a decentralized and distributed control paradigm [16]. The protocol requires that every node, asynchronously and independently, subdivides time into time slots with equal duration. At the beginning of each slot, the node has to choose whether to transmit or to switch to receiving mode. With a probability p the node starts transmitting a packet of data. At the end of transmission, the node has preemption over the channel, *i.e.*, if it chooses to perform another transmission (again with probability p) it is allowed to do it immediately: this guarantees a better throughput in case of bulk transmissions. If a node chooses to switch to receiving mode (with probability $1-p$), it remains in that mode either until the beginning of the next slot or until the channel becomes idle. The decision variable for each node is then its transmission probability, which may be different for different nodes. In a multiple node scenario, it is legitimate to model this quantity as a continuous variable evolving on a continuous time scale.

The simple decentralized algorithm proposed in [15] ensures convergence, full channel exploitation and fairness under the assumption of full connection (*i.e.*, if each node is connected to all the others). Under occlusions (*i.e.*, if the connection graph is not complete), fairness and even stability are not in general guaranteed. However, the control scheme can be rendered robustly stable with respect to topology switching by increasing the value of a certain parameter, at the price of compromising the performance in terms of channel exploitation.

In this paper we further investigate the control scheme in [15], by proving new theoretical properties and examining the trade–off between robust stability and performance.

- In the case of full connectivity, we show that the control variables satisfy the positivity constraints and the upper limits in terms of band exploitation. Fairness is proved by showing that the transmission rate variance is a Lyapunov–like function for the overall system.
- When occlusions are present, we show that the *robust* version of the scheme proposed in [15] ensures stability even in the presence of feedback delays, 0–transmission saturations (*i.e.*, the constraint that transmission rates cannot be negative) and topology switchings.
- We further consider the occlusion problem when connections are symmetric (*i.e.*, node i is (directly) linked to node j if and only if node j is (directly) linked to node i) and we show that stability can be assessed by standard eigenvalue techniques, even under network topology switchings, since it is related to the eigenvalue analysis of the adjacency matrix of an undirected graph, for which results are available in the literature.
- Also in the general case, results about the eigenvalue

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[1] Some of the presented results have been previously published in a preliminary form in [1].

analysis of the adjacency matrix can be exploited to find less stringent robustness requirements, which allow a better performance in terms of band exploitation. We identify a worst–case topology (bipartite graph) for which fast switchings can be treated.

- We tackle systems with multiple channels, in which nodes can switch among channels, based again on a fully decentralized decision. We show that the dynamics asymptotically converges to the equal exploitation of each channel.

Decentralized algorithms have been proposed for power control in wireless communication networks, such as the Foschini–Miljanic algorithm [17], see also the recent [18], [19]. Here, conversely, we are concerned with bandwidth allocation. The adopted approach is also related to consensus–oriented control [20], which is currently receiving attention especially in communication control problems [21], [22], [23], [24], [25], [26]. The essential difference is that here we do not focus on agreement among agents, but mainly on full band exploitation; however, we would like to ensure a "fair" allocation of resources, *i.e.*, an equal bandwidth available for each node. The eigenvalue analysis is similar in spirit, but we will be mainly interested in the smallest eigenvalue of the adjacency matrix, while the consensus is mainly influenced by the second smallest eigenvalue of the Laplacian matrix.

A. Description of the control problem

We consider the problem of n transmitting nodes, sharing a common communication channel of known bandwidth ρ. Each node autonomously decides its transmission rate based exclusively on its own rate and on the aggregate transmission rate of the other nodes. We define the variables:

- $x_i(t) \in \mathbb{R}$: message rate of the i-th node, $i = 1, \dots, n$;
- $y(t) = \sum_{i=1}^{n} x_i(t) \in \mathbb{R}$: total message rate;
- $u_i(t)$: control signal at the i-th node;
- $z_i(t) = \sum_{j \neq i} x_j(t) = y(t) - x_i(t)$: aggregate message rate of the nodes complementary to the i-th node.

Each node autonomously controls its transmission rate according to the following equation:

$$\dot{x}_i(t) = u_i(t). \tag{1}$$

The overall control goal is the ideal steady–state condition

$$\lim_{t \to \infty} x_i(t) = \frac{\rho}{n}, \quad \forall i, \tag{2}$$

which implies fairness and full band exploitation (since $\lim_{t \to \infty} y(t) = \rho$).

A control law of the form $u_i = u_i(x_i, z_i)$ would require full connection (*i.e.*, each node should be connected to all the others). Yet this is not always the case.

Definition 1: Let $\mathcal{C}_i$, for $i = 1, \dots, n$, denote the set that indexes the nodes connected to node i. The *connectivity degree*, c_i, is the number of elements of $\mathcal{C}_i$. Also, let

$$w_i(t) = \sum_{j \in \mathcal{C}_i} x_j(t)$$

denote the aggregate transmission rate of the nodes in $\mathcal{C}_i$. ◊

If the network is not fully connected, each node can rely on w_i only, hence the control has to be of the form $u_i = u_i(x_i, w_i)$. We will consider the following linear law

$$u_i(t) = -\alpha(1 + \mu_i)x_i(t) - \alpha w_i(t) + \alpha\rho, \tag{3}$$

where α and μ_i are positive parameters.

In [15] it is shown that in the case of full connection, *i.e.*, when $w_i(t) = z_i(t)$, the control law (3) renders system (1) asymptotically stable and that, if $\mu_i = \mu$ for all i, then

$$\lim_{t \to \infty} x_i(t) = \frac{\rho}{n + \mu}. \tag{4}$$

Therefore, in the case of full connection, (2) can be satisfied up to an arbitrarily small tolerance.

In most of the paper we will work under the following assumption.

Assumption 1: The protocol is common to all the nodes: $\mu_i = \mu$.

The assumption will be temporarily removed in Subsection II-D to discuss a possible adaptive mechanism for $\mu_i(t)$, which (though common to all nodes) can lead to different gains.

II. Analysis of the control scheme

The system described by (1) and (3) can be represented as

$$\dot{x}(t) = Ax(t) + \alpha\bar{\mathbf{1}}\rho \tag{5}$$

where $\bar{\mathbf{1}} \doteq [\, 1 \; 1 \; \dots \; 1 \,]^\top$ and

$$A = \alpha \begin{bmatrix} -(1+\mu) & -\delta_{12} & -\delta_{13} & \cdots & -\delta_{1n} \\ -\delta_{21} & -(1+\mu) & -\delta_{23} & \cdots & -\delta_{2n} \\ -\delta_{31} & -\delta_{32} & -(1+\mu) & \cdots & -\delta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\delta_{n1} & -\delta_{n2} & -\delta_{n3} & \cdots & -(1+\mu) \end{bmatrix},$$

with $\delta_{ij} = 1$ if node j is linked to node i, $\delta_{ij} = 0$ otherwise. We can compactly write

$$A = -\alpha[(\mu + 1)I + \Delta],$$

where Δ is the adjacency matrix associated with the system graph.

A. Some theoretical results about the control scheme under full connection

Under full connection assumption, the state matrix has the form $A = -\alpha[\mu I + O]$, where O is a matrix of ones. Its eigenvalues, $-\alpha(n + \mu)$ and $-\alpha\mu$,[2] are real and negative for all $\mu > 0$.

The state of system (5) always remains positive and satisfies the band upper limit.

Proposition 1: For any $\mu > 0$, the simplex

$$\mathcal{S} = \left\{ x : \; x_i \geq 0, \; \sum_{i=1}^{n} x_i \leq \rho \right\} \tag{6}$$

is positively invariant: if $x(0) \in \mathcal{S}$, then the solution $x(t)$ of (5) remains in $\mathcal{S}$ for all $t \geq 0$.

[2] Note that O has rank 1 and its eigenvalues are n, with multiplicity 1, and 0, with multiplicity $n - 1$.

Proof: The derivative of the total message rate $y(t)$ is

$$\dot{y} = -\alpha \sum_{j=1}^{n} [(1+\mu)x_j(t) + z_j(t) - \rho] = -\alpha[(n+\mu)y - \rho n] .$$

The time–variation of y associated with the initial condition $y(0)$ is then

$$y(t) = \left(y(0) - \frac{\rho n}{n+\mu} \right) e^{-\alpha(n+\mu)t} + \frac{\rho n}{n+\mu} .$$

Therefore

$$\lim_{t\to\infty} y(t) = \frac{\rho n}{n+\mu} < \rho ,$$

which implies that, when μ is small, a large quote of the band is exploited. The bound $y \leq \rho$ is never violated if $y(0) \leq \rho$, because, when $y = \rho$, we have $\dot{y} < 0$. To show that $x_i \geq 0$ is never violated as well, suppose that $x_i = 0$ and $x_j \geq 0$, for $i \neq j$. Then, equation (3) yields

$$\dot{x}_i = -\alpha[z_i - \rho] \geq 0,$$

where the inequality holds since $z_i \leq y \leq \rho$. Therefore, x_i cannot become negative and the claim is proved. ■

To show that the system monotonically tends to fairness, we define its variance[3] to be

$$V = \sum_{i=1}^{n} (x_i - y/n)^2 \tag{7}$$

and we prove it is a Lyapunov–like function for system (5).

Proposition 2: The variance function (7) is monotonically decreasing and tends to zero.

Proof: Exploiting the expression of $\dot{y}$ provided in the proof of Proposition 1, we have

$$\begin{aligned} \dot{V} &= 2\sum_{i=1}^{n}(x_i - y/n)(\dot{x}_i - \dot{y}/n)) \\ &= -2\alpha \sum_{i=1}^{n}(x_i - y/n)\left[\mu x_i + y - \rho - \frac{(n+\mu)y - \rho n}{n}\right] \\ &= -2\alpha \sum_{i=1}^{n}(x_i - y/n)(x_i - y/n)\mu = -2\alpha\mu V. \end{aligned}$$

Then $V(t) = e^{-2\alpha\mu t}V(0)$ and $\lim_{t\to\infty} V(t) = 0$. ■

Note that the total message rate is associated with the fast mode $-\alpha(n+\mu)$, while the variance is associated with the slow mode $-\alpha\mu$. As a consequence, y converges quickly to its equilibrium value $\rho n/(n+\mu)$, while the variance converges slowly to zero, especially if μ is small. However, the control law is designed so that, to achieve fairness, it is not necessary to wait for the transmission rates to reach their steady state; the convergence of the variance to zero, *i.e.*, the convergence of the system to a fair occupancy of the channel, is independent of the convergence of each channel to its steady state.

It can also be seen that the behaviour of the system is robust with respect to the insertion of new nodes. In fact, assume that, at $t = t_0$, k new nodes are added with zero initial transmission rate. The state $\hat{x}$ of the new system has dimension $n + k$, with initial condition $\hat{x}_i(t_0) = x_i(t_0)$, for $i = 1, \ldots, n$, and $\hat{x}_i(t_0) = 0$, for $i = n+1, \ldots, n+k$. As long as the old state $x(t_0)$ is in $\mathcal{S}$, the new state $\hat{x}$ at time t_0^+ will be in $\mathcal{S}_{n+k}$, the analogous set defined in the augmented state space. So no boundary violations are possible.

[3] Note that $y/n = \sum_i x_i/n$ corresponds to the average

B. Robustness under occlusions and communication delays

Under occlusions, stability is no longer ensured for any value of $\mu > 0$; however, it is preserved as long as the parameter μ is large enough.

Theorem 1: ([15]) The control law (3) renders system (1) asymptotically stable (for arbitrary $\mathcal{C}_i$), provided that $\mu > \max_i(c_i)$. Stability is ensured even under switching (namely, when the sets $\mathcal{C}_i$ can change arbitrarily in time). □

Stability is ensured if $\mu > \max_i(c_i)$; however, c_i are typically unknown. Since $c_i \leq n-1$, we can rely on the bound

$$\mu > n - 1. \tag{8}$$

As it will be shown later, this bound is penalizing in terms of performance. Conversely, we now show that large values of μ can ensure robust stability under communication delays. Suppose that each node receives information about the transmission rates of its adjacent nodes (the nodes in $\mathcal{C}_i$) with delays. In this case, equation (3) becomes

$$u_i(t) = -\alpha(1+\mu)x_i(t) - \alpha \sum_{j\in\mathcal{C}_i} x_j(t-\tau_{ij}) + \alpha\rho, \tag{9}$$

where τ_{ij} are positive constants. We assume that

$$0 \leq \tau_{ij} \leq \tau$$

for some *unknown* bound τ. As proved in the following theorem, condition (8) ensures robustness with respect to measurement delays.

Theorem 2: System (1) with the control law (9) is exponentially stable for any finite value of τ, provided that condition (8) is fulfilled. □

Proof: Being the system linear, we can study stability for $\rho = 0$. Consider the auxiliary variables

$$\xi_i(t) \doteq e^{\beta t}x_i(t), \quad i = 1, \ldots, n,$$

with $\beta > 0$. We prove that if β is small enough, and in particular $\beta < \alpha(1+\mu)$, then $\xi_i(t)$ is bounded for all t and for any initial condition $x_i(\sigma)$ with $\sigma \in [-\tau, 0]$. Boundedness of $\xi_i(t)$ implies exponential stability, since $x_i(t) = e^{-\beta t}\xi_i(t)$.

By computing the time–derivative of $\xi_i(t)$, we obtain

$$\begin{aligned} \dot{\xi}_i(t) &= e^{\beta t}\dot{x}_i(t) + \beta e^{\beta t}x_i(t) \\ &= -\alpha(1+\mu)e^{\beta t}x_i(t) \\ &\quad -\alpha \sum_{j\in\mathcal{C}_i} e^{\beta(t-\tau_{ij})}x_j(t-\tau_{ij})e^{\beta\tau_{ij}} + \beta e^{\beta t}x_i(t) \\ &= -\alpha\left(1+\mu-\frac{\beta}{\alpha}\right)\xi_i(t) - \alpha\sum_{j\in\mathcal{C}_i}\xi_j(t-\tau_{ij})e^{\beta\tau_{ij}}. \end{aligned} \tag{10}$$

To prove boundedness of $\xi_i(t)$ for all i and for all t, assume that for some $\kappa > 0$ and for some $\bar{t}$, $|\xi_i(\bar{\sigma})| \leq \kappa$ for all $i = 1, \ldots, n$ and for all $\bar{\sigma} \in [\bar{t} - \tau, \bar{t}]$. We prove that such a bound is respected for $t \geq \bar{t}$. By contradiction, assume that, for some $l \in \{1, \ldots, n\}$ and for some $\hat{t} \geq \bar{t}$, $|\xi_l(\hat{t})| = \kappa$ and

$\xi_l(\hat{t})\dot{\xi}_l(\hat{t}) > 0$, so that $|\xi_l(\hat{t}+\epsilon)| > \kappa$ for some $\epsilon > 0$. Consider the case $\xi_l(\hat{t}) = \kappa$ and $\dot{\xi}_l(\hat{t}) > 0$; if $\beta < \alpha(1+\mu)$, we have $(1+\mu-\beta/\alpha) > 0$ and, from (10), we obtain

$$\begin{aligned}\dot{\xi}_l(\hat{t}) &\leq -\alpha\left(1+\mu-\frac{\beta}{\alpha}\right)\kappa+\alpha\sum_{j\in\mathcal{C}_l}|\xi_j(\hat{t}-\tau_{lj})|e^{\beta\tau_{lj}}\\ &\leq \kappa\left[-\alpha\left(1+\mu-\frac{\beta}{\alpha}\right)+\alpha(n-1)e^{\beta\tau}\right]\\ &\leq \kappa\{-\alpha-\alpha[\mu-(n-1)e^{\beta\tau}]+\beta\}.\end{aligned}$$

For β small enough, $\dot{\xi}_l(\hat{t}) < 0$, which is in contradiction with the assumption $\dot{\xi}_l(\hat{t}) > 0$. Therefore, the upper bound $\xi_l(t) \leq \kappa$ cannot be violated. The case with $\xi_l(\hat{t}) = -\kappa$ and $\dot{\xi}_l(\hat{t}) < 0$ can be treated analogously. ■

C. Positivity constraints under switching topology

Transmission rates are subject to positivity constraints, $x_i \geq 0$, which are satisfied by the above control scheme, provided that μ is taken according to (8), as shown next.

Proposition 3: The control law (3) with $\mu > n-1$ ensures that the simplex set $\mathcal{S}$ (6) is robustly (*i.e.*, under arbitrary switching) positively invariant. □

Proof: We first show that x_i cannot become negative. Let $x_i = 0$ for some i and let all the remaining coordinates be such that $y \leq \rho$. Then

$$\dot{x}_i = -\alpha\sum_{k\in\mathcal{C}_i}x_k+\alpha\rho \geq -\alpha\rho+\alpha\rho = 0.$$

Therefore, x_i cannot become negative. Second, we show that y cannot exceed ρ. Let $y = \rho$. Then the following chain of equalities and inequalities holds:

$$\dot{y} = \sum_{i=1}^{n}\dot{x}_i = \sum_{i=1}^{n}[-\alpha(1+\mu)x_i-\alpha w_i+\alpha\rho] =$$
$$-\alpha(1+\mu)y-\alpha\sum_{i=1}^{n}\left(\sum_{k\in C_i}x_k\right)+\alpha\rho n \leq -\alpha(1+\mu-n)\rho < 0\,.$$

Therefore, y cannot go above ρ. ■

As a consequence of condition $\mu > n-1$, the total message rate at the equilibrium, $\bar{y}$, is strictly less than ρ: $\bar{y} < \rho$. This happens because, when $y = \rho$, we have $\dot{y} < 0$, hence ρ cannot be an equilibrium. Therefore, the i-th message rate at the equilibrium is

$$\bar{x}_i = \frac{-\sum_{j\in\mathcal{C}_i}\bar{x}_j+\rho}{1+\mu} > 0, \tag{11}$$

as it can be checked by plugging (11) into the system equations. We have the following corollary.

Corollary 1: If $\mu > n-1$, then for any fixed matrix Δ there exists an equilibrium point $\bar{x}_i(\Delta)$ inside the set $\mathcal{S}$, which is positive component–wise. □

Therefore, in view of Theorem 1, the solution of system (1), with u_i as in (3) and $\mu > n-1$, converges to the positive equilibrium $\bar{x}_i$.

Remark 1: In general, for values of μ smaller than $n-1$ or for initial conditions outside $\mathcal{S}$, the proposed control law (3) may drive x_i to negative values. In this case, to avoid negative values, one has to saturate the control law in (1) according to the following rule:

$$\dot{x}_i(t) = \begin{cases} u_i(t) & \text{if} \quad x_i > 0,\\ \max\{0, u_i(t)\} & \text{if} \quad x_i = 0.\end{cases} \tag{12}$$

D. Gain adaptation

We can improve the scheme by allowing $\mu(t)$ to be time–varying in an adaptive way (high gain adaptation [27]). To this aim, we temporarily drop Assumption 1 and state the following.

Theorem 3: Consider the linear time–varying system governed by matrix

$$A(t) = -\alpha[(\Sigma(t)+I)+\Delta(t)],$$

where $\Sigma(t) = \text{diag}\{\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\}$ and the entries $\Delta_{ij}(t)$ of $\Delta(t)$ can switch in $\{0,1\}$. A value $\bar{\mu}$ exists such that, if $\mu_i(t) \geq \bar{\mu}$ for all $t > \bar{t}$ for some $\bar{t}$, then the system is asymptotically stable.

Proof: Denote the whole state of the network by $\mathbf{x} = (x_1 \ldots x_n)^\top$. The claim surely holds for any $\bar{\mu} > n-1$, since in this case the matrix is diagonally dominant for all $t > 0$, hence the system admits the piecewise–linear function

$$\Psi(\mathbf{x}) = \max_i\ |x_i| = \|\mathbf{x}\|_\infty$$

as a Lyapunov function [28], [29], [30], [31] (see [32], Proposition 4.33). However, the diagonal dominance could hold also for $\bar{\mu} \leq n-1$. ■

Remark 2: For the case of a constant topology and constant μ, stability could be proved through the Gershgorin circle theorem, which provides a bound for the eigenvalues. Yet, since we have found a common Lyapunov function independent of the considered topology and of the value of μ, the result holds even when topology switchings occur and when μ is time–varying.

Based on Theorem 3, an adaptive scheme, in which the value of μ_i is increased over time, can be designed as follows. Each node is initialized with a (small) value $\mu_i(0) > 0$. Under full connection, this would ensure stability. If instability is detected, for instance evidenced by transmission fluctuations, then each node independently starts rising its gain $\mu_i(t)$ according to the adaptive law

$$\dot{\mu}_i(t) = \begin{cases} \delta_i\,, & \text{if} \quad 0 \leq t \leq \theta_{s,i}\\ 0\,, & \text{if} \quad t > \theta_{s,i}\end{cases} \tag{13}$$

where $\delta_i > 0$, while $\theta_{s,i} \geq 0$ is a time at which instability is not detected anymore by node i. We show the effectiveness of this adaptive scheme with an example.

Example 1: To detect instability at each node, we monitor the derivative of the aggregate transmission rate $\dot{w}_i$, suitably filtered, and compare it with a threshold value ϵ. More precisely, we define the filtered signal $w_i^F(t) = \mathcal{L}^{-1}\left\{\frac{s}{s+\sigma}\mathcal{L}\{w_i(t)\}\right\}$, where $\mathcal{L}$ denotes the Laplace transform and σ is a positive constant, and we apply the rule (13): $\dot{\mu}_i(t) = \max\{0, \rho(|w_i^F(t)|-\epsilon)\}$ for some positive ρ.[4] Figure 1 reports the simulation results for a randomly generated graph

[4]In this way adaptation stops (*i.e.*, $\dot{\mu}_i = 0$) when $|w_i^F(t)| \leq \epsilon$.

with $n = 100$ and occlusion probability $\mathbb{P}[\Delta_{ij} = 0] = 0.3$. The initial values for the gain are $\mu_i(0) = 10$ for all i, while the values of the constants are $\epsilon = 0.05$, $\rho = 50$ and $\sigma = 1$. It can be noted that the steady state values of the gains remain well below the value $n-1 = 99$ used, as worst case value, in the proof of Thorem 3.

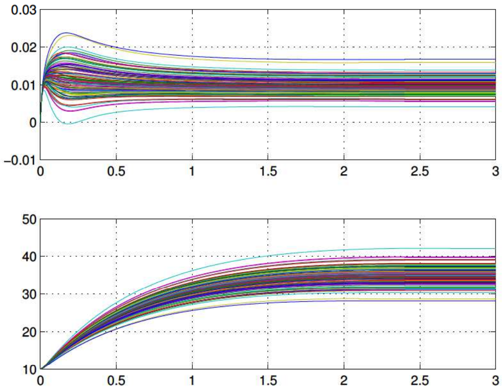

Fig. 1: An example of gain adaptation for $n = 100$. Top: transmission rates $x_i(t)$; bottom: gains $\mu_i(t)$.

III. Switching–robustness versus steady–state performance

In this section, we resume Assumption 1 and we investigate the trade–off in the choice of μ.

A. Band exploitation under uniform connectivity degree

We have shown that the control scheme is robust with respect to occlusions (partial connection), at the cost of a reduced band exploitation. The trade–off between robustness and performance can be investigated by considering an ideal case in which all the nodes have the same connectivity degree.

Proposition 4: Assume that $\mu > 0$ assures stability and that each node of the graph has the same connectivity degree, $c_i = c$. Then the total steady–state transmission rate is given by

$$\bar{y} = \frac{\rho n}{\mu + c + 1}\,. \tag{14}$$

□

Proof: If μ assures stability, then all the transmission rates admit a single steady–state value which, by symmetry, is the same for all nodes. Let $\bar{x}$ denote this value. The i-th equilibrium condition becomes

$$0 = -(\mu+1)\bar{x} - \sum_{k\in\mathcal{C}_i} \bar{x} + \rho\,,$$

hence

$$\bar{x} = \frac{\rho}{\mu + c + 1}\,.$$

Since $\bar{y} = \sum_{i=1}^{n} \bar{x}$, we obtain the above expression (14). ■

In the case of full connection ($c = n-1$) and assuming $\mu > n$ but not too large (for instance $\mu = n+1$), we obtain

$$\bar{y} = \frac{\rho n}{\mu + n} \approx \frac{\rho}{2}, \tag{15}$$

hence about half of the bandwidth is lost. Conversely, for low connectivity (c small compared to n), $\bar{y}$ is approximately equal to the whole bandwidth ρ.

B. Worst–case analysis

Expression (15) shows that the performance, in terms of bandwidth exploitation, might be compromised to ensure switching–robust stability. The analysis in this subsection will enlighten the role of the network topology, revealing that there is a *worst case*: the bipartite graph. We will also show that stronger results can be obtained for undirected graphs (*i.e.*, symmetric connections: the communication between a pair of nodes, if it occurs, is in both directions). In this case, time–varying (switching) and constant topologies are equivalent, and the problem reduces to an eigenvalue analysis. We begin precisely by investigating the case of symmetric connections, in which $\delta_{ij}(t) = \delta_{ji}(t)$ for all t, and we consider any family of topologies fulfilling some abstract requirement $\mathcal{R}$; the corresponding set of adjacency matrices is

$$\mathcal{T} = \{\Delta \;:\; \Delta \text{ fulfills } \mathcal{R}\}.$$

The closed–loop system that has to be considered is described by the equation

$$\dot{\mathbf{x}}(t) = -\alpha[(1+\mu)I + \Delta(t)]\mathbf{x}(t) + \alpha\rho\bar{\mathbf{1}}, \tag{16}$$

where $\Delta(t) \in \mathcal{T}$ for all t. Under these assumptions we have the following result.

Theorem 4: In the case of symmetric connections, the following conditions are equivalent:

i) any matrix $S = -[(1+\mu)I + \Delta]$ with $\Delta \in \mathcal{T}$ is Hurwitz (*i.e.*, has negative eigenvalues);
ii) the system is stable under arbitrary switching among the topologies in $\mathcal{T}$;
iii) for each fixed Δ, the system admits the Lyapunov function

$$\Psi(\mathbf{x} - \bar{\mathbf{x}}(\Delta)) = (\mathbf{x} - \bar{\mathbf{x}}(\Delta))^\top(\mathbf{x} - \bar{\mathbf{x}}(\Delta)),$$

where $\bar{\mathbf{x}}(\Delta)$ is the equilibrium value. □

Proof: The equivalence between ii) and iii) is guaranteed by the theory of common Lyapunov functions (see, for instance, [33], Theorem 2.1). ii) ⇒ i): if the system is stable under arbitrary switching among topologies, of course each matrix corresponding to a particular topology is Hurwitz stable. i) ⇒ ii): if any S is Hurwitz, the identity matrix I is a solution of the Lyapunov equation $S^\top X + XS = 2S$, in which the term $2S$ is negative definite (being S Hurwitz), hence we can provide a common Lyapunov function which ensures stability under arbitrary switching. ■

The following corollary points out the role of the smallest eigenvalue of the adjacency matrix.

Corollary 2: In the case of symmetric connections, stability of system (16) is ensured if and only if, denoting by λ_i, $i = 1, 2, \ldots, n$, the (obviously real) eigenvalues of Δ, we have

$$\lambda_i > -(1+\mu), \quad \forall\, i\,. \tag{17}$$

□

Remark 3: Equation (17) shows an interesting connection between the protocol performance (related to the choice of μ) and the topology of the network on which the protocol runs (related to the eigenvalues λ_i).

As a consequence of the above results, the stability problem for a network with symmetric connections, whose adjacency matrix switches among the matrices of a given family $\mathcal{T}$, may be reduced to the determination of the smallest eigenvalue of all the matrices of the family.

It is immediate to examine the two extreme cases, namely, the no–connection case ($\Delta = \mathbf{0}$), for which $\lambda_i = 0$ for all $i = 1, \ldots, n$, and the full connection case ($\Delta_{ij} = 1$, for all $i, j = 1, \ldots, n$, $i \neq j$, $\Delta_{ij} = 0$ otherwise). In this latter case, matrix A becomes $-\alpha[\mu I + O]$ and, since the eigenvalues of O are $\lambda_1 = n - 1$ and $\lambda_i = -1$ for $i \neq 1$, stability is ensured for any $\mu > 0$, as previously seen. The intermediate topologies are obviously much more interesting.

Since (17) holds for all eigenvalues provided it holds for the minimum, it is reasonable to investigate the worst case, *i.e.*, to give an answer to the following question.

Question: Given the dimension of the network, *i.e.*, the number n of nodes, which is the symmetric topology having the smallest eigenvalue?

We associate such a topology with the choice $\mu = \mu_{symm}$. To the best of our knowledge, the problem of characterizing the minimum eigenvalue of the adjacency matrix of a graph is still open, even though several works have been published on the topic [34], [35], [36], [37], [38], [39], [40], [41]. In the following, we present two particular cases that are easy to analyze and worth being discussed.

Proposition 5: Assume n even. The eigenvalues of matrix

$$\Delta = \begin{bmatrix} \mathbf{0} & O_{n/2} \\ O_{n/2} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{0} \in \mathbb{R}^{n/2 \times n/2}$ is a null matrix and $O_{n/2} \in \mathbb{R}^{n/2 \times n/2}$ is a ones matrix, are $\lambda_1 = -n/2$, $\lambda_2 = \cdots = \lambda_{n-1} = 0$, $\lambda_n = n/2$. □

Proof: The vectors $[\bar{\mathbf{1}}^\top \ \bar{\mathbf{1}}^\top]^\top$ and $[-\bar{\mathbf{1}}^\top \ \bar{\mathbf{1}}^\top]^\top$ are eigenvectors associated with the eigenvalues $n/2$ and $-n/2$, respectively. Since $\mathrm{rank}(\Delta) = 2$, the other eigenvalues are zero. ■

Proposition 5 together with (17) implies that a safe value of μ for symmetric topologies is

$$\mu_{symm} > \frac{n}{2} - 1\,. \tag{18}$$

The bipartite topology considered in Proposition 5 is indeed the worst case, as shown by the following result, hence the provided bound is tight.

Theorem 5: [41] If Δ is the adjacency matrix of a graph of order n, then the smallest eigenvalue of Δ, denoted by $\lambda_1(\Delta)$, satisfies the inequality[5]

$$\lambda_1(\Delta) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$$

and the equality holds if and only if the graph is bipartite into two subgraphs of orders $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.[6] □

The bound previously found for all (possibly non–symmetric) topologies, *i.e.*, $\mu > n - 1$, may be considered conservative, but it is not more than twice bigger than necessary.

Consider now the band exploitation in the ideal case when all nodes are connected. Using (4) and selecting for μ the minimum value fulfilling (18), namely, $\mu = n/2 - 1 + \epsilon$ for some (small) $\epsilon > 0$, we obtain the following common limit value for all the nodes:

$$\bar{x}_i = \frac{\rho}{n + \mu} = \frac{\rho}{n + \frac{n}{2} - 1 + \epsilon} = \frac{2\rho}{3n - 2 + 2\epsilon}\,. \tag{19}$$

If $\epsilon < 1$, we obtain $\bar{x}_i > 2\rho/3n$; hence, under symmetry assumption, the band exploitation is larger than the general value $\rho/2$ guaranteed by (15).

It is interesting to note that, also in the case of non–symmetric connections, we can provide the same bound for μ obtained in (18), as shown next.

Proposition 6: Stability of system (16) is ensured for any matrix Δ, possibly non–symmetric, if

$$\mu > \frac{n}{2} - 1\,. \tag{20}$$

□

Proof: Consider the matrix $S = -[(1 + \mu)I + \Delta]$ with $\Delta \in \mathcal{T}$ and the Lyapunov equation

$$X^\top + X = Q\,. \tag{21}$$

Stability is guaranteed if S is a solution of (21) for some negative definite matrix Q. We have

$$S^\top + S = -2(\mu + 1)I - (\Delta^\top + \Delta) = -2(\mu + 1)I + M,$$

where $M \doteq -(\Delta^\top + \Delta)$. Since $-M \in \mathbb{R}^{n \times n}$ is a symmetric matrix whose entries are in the interval $[0, 2]$, the results in [41], [42] apply and provide a lower bound for the smallest eigenvalue of $-M$: $\lambda_1(-M) \geq -n$. This implies that $\lambda_n(M) \leq n$, where $\lambda_n(M)$ is the largest eigenvalue of M. If $\mu > n/2 - 1$, then the matrix $Q = -2(\mu + 1)I + M$ is negative definite. ■

Remark 4: Note that the bound $\mu > n/2 - 1$ cannot be improved, since it is tight in the symmetric–connection case.

There are particular topologies that may be stabilized with very small values of μ, thus providing a good performance. This is the case, for instance, of topologies formed by isolated clusters. The isolated cluster topology consists of a finite set of sub–graphs, called *clusters*, which are (internally) fully connected, but are not connected to one another. This means that the corresponding state matrix is block–diagonal.

Proposition 7: Assume that the matrices in $\mathcal{T}$ correspond to topologies formed by isolated clusters. Then $\mu > 0$ assures stability of system (16).

[5] We denote by $\lfloor \cdot \rfloor$ and by $\lceil \cdot \rceil$ the floor and the ceiling functions, respectively.

[6] The argument about the eigenvalues holds because the matrix is symmetric.

Proof: The state matrices belonging to this class of topologies have the following structure:

$$-\alpha \begin{bmatrix} \mu I+O_1 & 0 & \cdots & 0 \\ 0 & \mu I+O_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu I+O_k \end{bmatrix},$$

where O_i are ones matrices of possibly different dimension. If $\mu > 0$, all these matrices have negative eigenvalues. ■

The previous proposition explains also why the simulations carried out in [15] evidence a very good performance even for small values of $\mu > 0$ in the cases in which topology changes preserving an isolated cluster structure are imposed.

The condition $\mu > n/2 - 1$ is less conservative than (8) and has been proved to be sufficient for stability. Yet it does not guarantee that all the transmission rates are positive at the equilibrium, as shown by the following example.

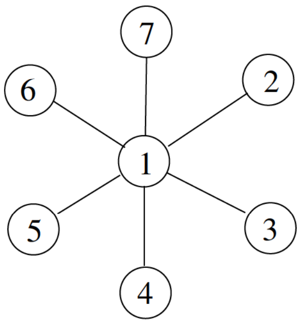

Fig. 2: An example of a graph admitting a stable equilibrium in which not all the transmission rates are positive.

Example 2: Consider the star graph in Figure 2, where node 1 is connected to nodes 2 to 7, which do not communicate among them. Consider system (16) with $\alpha = \rho = 1$. Direct calculations yield, for $\mu = 3$,

$$-S = (1+\mu)I + \Delta = \begin{pmatrix} 4 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 0 & \cdots & 0 \\ 1 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 4 \end{pmatrix},$$

from which $\bar{\mathbf{x}} = -S^{-1}\bar{\mathbf{1}} = [-0.2\ 0.3\ 0.3\ \cdots\ 0.3]^\top$. Clearly in this case the central node would not transmit, due to the lower saturation, and the total bandwidth exploited by the external nodes would be much greater than the allowable $\rho = 1$. This example points out that, for completely unbalanced situations, a value $\mu > n$ is desirable. However, unbalanced situations as in Figure 2 are unrealistic.

C. Statistical evaluation of the minimum eigenvalue

To further investigate the connection between the topology and the minimum eigenvalue of the adjacency matrix of random graphs [43], we have considered sets of Erdős–Rényi graphs [44], [45] $G(n, p)$ with $n = 20,\ 40,\ 60,\ 80,\ 100$ (one set for each order), where the connection between any couple of nodes is occluded with a probability $q = 1 - p \in [0, 1]$ and connections are symmetric (if i is connected to j, j is connected to i). We have computed the average minimum eigenvalue, whose absolute value is plotted in Figure 3 (with dots) as a function of q. The simulations are consistent with

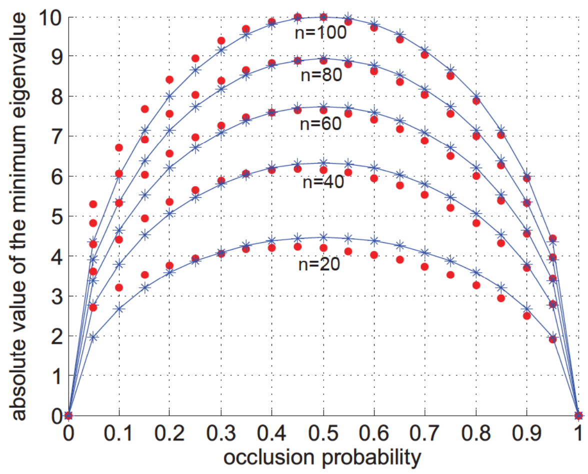

Fig. 3: Red dots: absolute value of the average minimum eigenvalue for random Erdős–Rényi graphs (symmetric connection) as a function of the occlusion probability; blue line with asterisks: theoretical estimate.

the theoretical results in [46] (in particular with Theorem 5.1, referred to [47]), which provide the bound $-\lambda_n \leq (2+o(1))\sqrt{np(1-p)}$, where $o(1)$ is a first–order infinitesimal for $n \to \infty$. The theoretical estimate, shown by the solid line with asterisks in Figure 3, holds in principle for very large graphs (see [46] for details), but it is approximately matched as well by the graphs we have considered. As expected, the minimum eigenvalue is zero in the extreme cases of no connection (isolated nodes) and fully connected graphs, while its minimum value occurs when the connection probability is about 0.5. However, the average minimum eigenvalue is much less than that obtained for the bipartite graph: for $n = 100$, for instance, the average minimum eigenvalue in the case of symmetric connections is minimized in the case of occlusion probability 0.5 and corresponds to -10, while the minimum eigenvalue for a bipartite graph is -50.

Stronger results under switching topology are expected if one imposes dwell–time constraint, *i.e.*, the requirement that the configuration of the system must remain constant for a minimum time–interval [48], [49] (see [50] for a survey). The only question is how reasonable is the assumption of a dwell–time in systems that can be of a very large scale.

IV. The Multiple Channel Case

We now consider the case in which, instead of a single channel, there are N independent channels with the same bandwidth ρ. We assume $\rho = 1$ for simplicity ($\rho \neq 1$ would be just a scaling factor). Each node has the knowledge of the existence of all the channels, but it does not have any information about the number of nodes transmitting in each of them or about the level of occupancy. Each node may decide to jump randomly from one channel to another on the basis of its current transmission rate. The approach proposed here is based on a decentralized resource allocation (see for instance [51] and references therein). We assume a time–scale separation: the process of deciding whether to remain in the current channel or to jump to another channel is much slower than the process of adjusting the transmission rate within a channel according to the control law described above. Typically each node is expected to wait until it reaches its steady–state before considering the possibility of jumping. Hence, the dynamics of jumps can be modeled in its own time–scale, slower than the time–scale of the processes considered in the previous sections. We assume that the jump probability, denoted by $p(x_i)$, is a strictly decreasing function of the transmission rate x_i. Suppose, to begin, that it is affine, namely, that

$$p(x_i) = p_0(1 - x_i)\,,$$

where p_0 is a constant such that $p_0 \in [0, 1]$. A reasonable choice for p_0 is $p_0 = 1$, so that when the transmission rate is zero, the node jumps with probability one. On the other hand, also the choice of a value less than one may be justified, with the aim of "giving a chance" to the current channel even if it is fully occupied. However, the actual value of p_0 is not crucial for the following reasoning and may be left unspecified. Since we have assumed that all the channels have the same bandwidth and that the nodes do not have information about the level of occupancy of the other channels, we assume also that, when a node decides to jump, it may choose with the same probability any other channel.

Given a channel $\mathcal{C}$ used by m nodes, we define the total channel message rate as $y_{\mathcal{C}} = \sum_{i=1}^{m} x_i$. The *migration flow* from channel $\mathcal{C}$, *i.e.*, the traffic leaving $\mathcal{C}$ and moving to other channels, is

$$\begin{aligned}\sum_{i=1}^{m} p_0(1 - x_i) &= p_0[m - y_{\mathcal{C}}] \simeq p_0\left[m - \frac{m}{m + \mu + 1}\right]\\ &= p_0[m - 1] + p_0\left[\frac{\mu + 1}{m + \mu + 1}\right] \simeq p_0 m\,,\end{aligned}$$

where we have introduced two approximations: we have considered the steady state value of $y_{\mathcal{C}}$ (cf. equation (15)) and we have assumed m large.

In a more general context, the probability may be not affine; however, it is reasonable to assume that the transmission rate associated with each node in a channel is a decreasing function of the number of nodes using the channel (the higher the number of nodes, the lower the bandwidth that can be occupied by each of them). Henceforth we assume that the migration flow from a channel is generally given by a strictly increasing function of the number of nodes using the channel (the higher the number of nodes, the higher the probability that some of them leave the channel). Moreover, even though the number of nodes in a channel may be only discrete, for large numbers of nodes it is allowable, and much easier, to associate it with a continuous variable. Let ν_k denote this number associated with the channel k, and let $\phi(\nu_k(t))$, for a function $\phi : \mathbb{R}_+ \to \mathbb{R}^+$, denote the number of nodes leaving the k-th channel at time t. For the aim of the following results, we choose ϕ to be even.

The $\phi(\nu_k(t))$ nodes leaving the k-th channel distribute uniformly among the other channels; as a consequence, the amount of nodes (leaving the k-th channel and) entering, say, the j-th one is $\phi(\nu_k(t))/(N-1)$. Hence the overall time–variation of the number of nodes using the k-th channel is

$$\dot{\nu}_k(t) = -\phi(\nu_k(t)) + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\phi(\nu_j(t))}{N-1}$$

Introducing the notations

$$\nu = [\nu_1\ \nu_2\ \ldots\ \nu_N]^\top\,,\ \Phi(\nu) = [\phi(\nu_1)\ \phi(\nu_2)\ \ldots\ \phi(\nu_N)]^\top\,,$$

the node distribution dynamics can be represented by

$$\dot{\nu}(t) = \Pi\,\Phi(\nu), \tag{22}$$

where Π is the symmetric matrix

$$\Pi = \begin{bmatrix} -1 & \frac{1}{N-1} & \frac{1}{N-1} & \cdots & \frac{1}{N-1}\\ \frac{1}{N-1} & -1 & \frac{1}{N-1} & \cdots & \frac{1}{N-1}\\ \frac{1}{N-1} & \frac{1}{N-1} & -1 & \cdots & \frac{1}{N-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N-1} & -1 & \cdots & \frac{1}{N-1} & -1 \end{bmatrix}.$$

Note that this equation implies that no node leaves the system. Indeed, the total number of nodes can be expressed as

$$n(t) = \bar{\mathbf{1}}^\top \nu(t) \tag{23}$$

and hence

$$\frac{d}{dt}n(t) = \bar{\mathbf{1}}^\top \dot{\nu}(t) = \bar{\mathbf{1}}^\top \Pi \Phi(\nu) = 0\,, \tag{24}$$

where the last equality follows from the structure of the matrix Π. We now show that, asymptotically, the node distribution among the channels is uniform.

Theorem 6: The trajectory $\nu(t)$ asymptotically converges to the average value $\bar{\nu} = n/N\bar{\mathbf{1}}$, *i.e.*, $\nu_k = n/N$ for all k. □

Proof: Let $\psi : [0, +\infty) \to \mathbb{R}$ be the function defined by

$$\psi(\xi) = \int_0^\xi \phi(\eta)d\eta\,,$$

which is zero for $\xi = 0$ and strictly positive increasing for $\xi > 0$. Consider the candidate Lyapunov function

$$\Psi(\nu) = \sum_{k=1}^{N} \psi(\nu_k)\,.$$

Ψ is co–positive (i.e. positive for $\nu_k \geq 0$) and its time–derivative is

$$\dot{\Psi}(\nu) = \Phi(\nu)^\top \Pi \Phi(\nu) \leq 0\,.$$

In view of the La Salle principle, the trajectory ν converges to the set $\mathcal{N} = \{\nu : \dot{\Psi}(\nu) = 0\}$. Matrix Π is negative semi–definite and has rank $N-1$. Its kernel is given by the set of vectors whose components are all equal, *i.e.*, $\Pi v = 0$ (or $v^\top \Pi = 0$ or $v^\top \Pi v = 0$) if and only if $v = \lambda\bar{\mathbf{1}}$ for some real λ. Since the components $\phi(\nu_k)$ of $\Phi(\nu)$ are invertible, $\nu(t)$ converges to the set $\mathcal{N} = \{\nu : \nu_1 = \nu_2 = \cdots = \nu_N\}$. On the other hand, in view of (23)-(24) the common value is $\nu_i = n/N$. ■

Remark 5: The previous result is due to the doubly–stochastic nature of matrix Π. We point out that channel commutation can be subject to constraints. For instance, a node could be allowed to jump only from a channel to one of the two channels which are adjacent in frequency (jumping "up" and "down"). If the probability of jumping up and that of jumping down are the same, then the resulting matrix Π has a tridiagonal (and still doubly–stochastic) form:

$$\Pi_{fo} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} \end{bmatrix}.$$

Using the same arguments as before, we can conclude that the asymptotic distribution is uniform.

V. Implementation and Example

In order to be effective, the provided scheme must be of simple implementation. To this aim, some details have to be considered. First of all, the scheme has to be digitally implemented and a sampling time has to be fixed. We assume for brevity that the sampling time is approximately the same time which is necessary for the transmission of a packet.

A fundamental aspect in our method is that the value of the variable x_j cannot be transmitted (by the j-th node) and hence each node must estimate the aggregate transmission rate w_i of the nodes in $\mathcal{C}_i$ (see Section I) which cannot be measured directly. An accurate way to estimate w_i is to count the packets transmitted by the other nodes and to take their average in a certain time–window of length τ_w:

$$\hat{w}_i = \frac{1}{\tau_w} \sum_{j \in \mathcal{C}_i} q_j \,,$$

where q_j is the number of packets transmitted by the (adjacent) node j and received by node i. The resulting discrete–time scheme, for $\alpha = 1$, $\rho = 1$ and for a sampling period of τ is

$$x_i((k+1)\tau) = x_i(k\tau) + \tau[-(\mu+1)x_i(k\tau) - \hat{w}_i + 1] \,.$$

Note that the value of τ_w should be a multiple of τ and that the averaging operation introduces a delay in the process.

To illustrate the results of the method, we have considered a system with $n = 300$ nodes and $N = 3$ channels. The nodes are spatially randomly distributed on a square with edge 1000 meters. The case of a partial connection is simulated by assuming that each node can receive packets transmitted by other nodes within the maximum distance of 500 meters.

We have assumed that the distribution of the nodes among the channels is eventually uniform (about 100 nodes per channel) and we have chosen $\mu = 110$. We have performed a simulation in which initially all the nodes use channel 1. The simulation length is 0.5 seconds, with a sampling period of 0.001 seconds. The desired transmission rate is 1 packet for each time sample. At each sampling time, each node jumps to another channel with probability $p_0(1 - x_i)$, with $p_0 = 5 \times 10^{-2}$.

Figure 4 represents packet transmission with respect to time, which is uniformly distributed among the nodes. In Figure 5 we see that, after a transient of about 0.1 seconds, the nodes become almost equally distributed among the three channels.

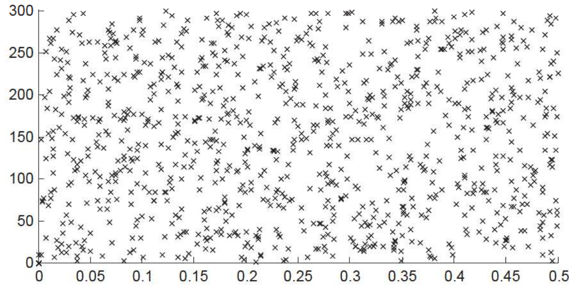

Fig. 4: Packet transmission vs. time (seconds): transmitting nodes are marked with 'x'.

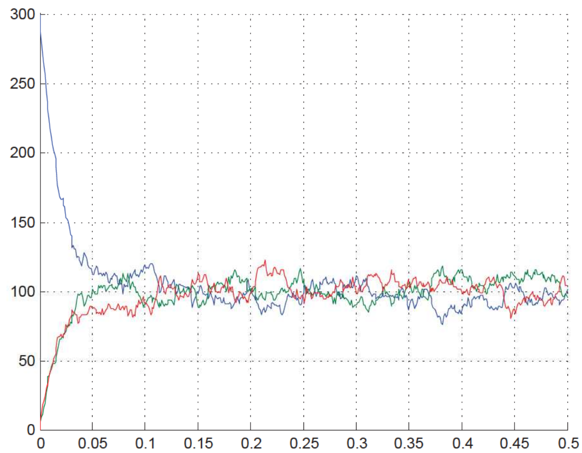

Fig. 5: The time evolution of node distribution among the available channels: number of nodes in each channel vs. time (seconds).

The average transmission rate of the whole system is 1.87 packets per sampling time, thus we have $1.87/3 = 0.62$ packets per channel, which is clearly smaller than the ideal value of 1 packet. This is the price to pay for a high value of μ, chosen to robustify the system. With the less conservative value $\mu = 70$, the average transmission rate, obtained in simulations whose detailed results are not shown, is larger, *i.e.*, 2.59 packets for the overall system and 0.86 packets per channel.

VI. Conclusions

We have analyzed the property of a distributed algorithm for a set of nodes sharing the same channel. We have shown that the algorithm is robust against occlusions, delays, 0–saturation and switching, provided that a certain parameter is taken large enough. Bounds are provided. Under symmetry assumptions, the parameter has to be larger than the absolute value of the smallest eigenvalue of the adjacency matrix of the network. We have extended the method to the scenario of multiple channels, showing that a fair situation is reached not only with respect to the transmission rate of each node in each single channel, but also with respect to the number of nodes per channel.

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