Compartmental flow control: decentralization, robustness and optimality

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Abstract

We consider the flow control problem for a general class of compartmental nonlinear systems, which can be associated with a graph whose nodes represent subsystems with their own internal dynamics, and whose arcs represent flow links among them. We consider a network–decentralized control: each agent controls a link between two nodes and decides its actions based on the states of these nodes only. We first provide general necessary and sufficient stabilizability conditions, proving that suitable network–decentralized strategies assure robust stability. We also show that, if all the subsystems at the nodes are marginally stable, a proper network–decentralized strategy asymptotically assures the minimum–norm flow, without requiring communication among agents.

Key words: Compartmental systems; Control of networks; Network–decentralized control; Optimality.

1 Introduction

The control of large networks is relevant in many contexts, including data transmission (Moss & Segall 1978; Moreno & Papageorgiou 1995; Ephremides & Verdú 1989; Iftar & Davison 1990, 2002), flow networks (Bauso, Blanchini, & Pesenti 2010; Atamturk & Zhang 2007; Ordóñez & Zhao 2007; Wei & van der Schaft 2013; Danielson, Borrelli, Oliver, Anderson, & Phillips 2013), inventory and production systems (Bertsimas & Thiele 2006; Blanchini, Rinaldi, & Ukovich 1997; Blanchini, Miani, & Ukovich 2000; Boukas, Yang, & Zhang 1995; Sarimveis, Patrinos, Tarantilis, & Kiranoudis 2008; Silver & Peterson 1985), water distribution networks (Larson & Keckler 1969; Bauso, Blanchini, Giarré, & Pesenti 2013), transportation (Ataslar & Iftar 1998; Mudchanatongsuk, Ordóñez, & Liu 2008) and traffic networks (Iftar 1996, 1999). Global communication is often impossible in large networks; hence, control agents must act based on *locally* available information. We say that a controller is *network–decentralized* if the control is actuated by agents: 1) each governing a network link (arc) that affects two subsystems (nodes of a graph), and 2) each making its decision exclusively based on local information (*i.e.*, on the states of the nodes to which it is directly connected). This type of control was considered by Iftar & Davison (1990, 2002); Iftar (1999); Blanchini, Miani, & Ukovich (2000) for buffer systems, where the subsystems are first–order integrators; for this class of systems, as shown by Bauso, Blanchini, Giarré, & Pesenti (2013), a proper constrained decentralized control law assures asymptotic optimality in the minimum–norm sense. A network–decentralized control is proposed by Blanchini, Franco, & Giordano (2013, 2015) for linear systems where nodes represent non–interacting subsystems, coupled by the control action; it is shown that, if the subsystems do not share unstable eigenvalues, any stabilizable system can be stabilized by a decentralized control.

In this paper we consider the problem of stabilization for a more general class of compartmental nonlinear systems, possibly interacting, in the presence of an external uncontrolled flow. Compartmental systems (Jacquez & Simon 1993) are monotone (Angeli & Sontag 2003; Smith 2008; Chisci & Falugi 2006) and are fundamental in many flow control problems. Our general approach takes into account decentralization, control constraints and robustness (*i.e.*, effectiveness of the control regardless of the system parameters). The main contributions of the paper are the following.

- We formulate the network–decentralized control problem for a broad class of nonlinear compartmental systems and we find a stabilizing network–decentralized control.
- Under flow constraints, we provide necessary and sufficient stabilizability conditions, in terms of network connectivity, adopting the saturated control proposed by Bauso, Blanchini, Giarré, & Pesenti (2013).
- We show that such a control, applied *tout court*, may not be stabilizing if the steady state corresponding to the current demand is not known.
- For a vast class of compartmental systems, we propose a modification of the strategy by Bauso, Blanchini, Giarré, & Pesenti (2013), assuring that the controlled flow converges to the optimum (minimum Euclidean norm), without requiring communication among agents.
- We investigate the special case (preliminarily presented by Blanchini, Giordano, & Montessoro (2014)) in which the nodes represent linear subsystems with an internal flow splitting represented by Markov chains, and we propose, as an example, a problem of data flow control.

2 Model description

We consider a class of models of the form

$$\dot{x}(t) = Sg^*(x(t)) + Rh^*(x(t)) + Bu(t) + d(t), \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^n$ is an exogenous signal. We refer to $u_j(t) \in \mathbb{R}$ as control *agents* and to $d(t)$ as *demand*. A graph $\mathcal{N}$ with n nodes can be associated with the system; we assume that S, R and B are incidence matrices for $\mathcal{N}$: each of their columns has either two non–zero entries, equal to 1 and -1, or a single non–zero entry, equal either to 1 or to -1.

Vector functions g^* and h^* represent flows between two nodes within the system (application examples include flowing of data, fluids, or currents) and have a different physical meaning: g–type flows, associated with g^*, depend on the difference between the corresponding states, while h–type flows, associated with h^*, depend on the state of the starting node only. For instance, in fluid systems, g–type flows between tanks depend on the fluid level in both tanks, while h–type flows depend on the fluid level in the upper tank only (the two cases are illustrated in Fig. 1). Formally, we have:

- $g_j^* = g_j^*(x_k - x_l)$, where $S_{kj} = -1$ and $S_{lj} = 1$;
- $h_j^* = h_j^*(x_k)$, where $R_{kj} = -1$.

Denoting by M_j the jth column of a matrix M, we can write

$$g_j^* = g_j^*(-S_j^\top x) \quad \text{and} \quad h_j^* = h_j^*(-\tilde{R}_j^\top x),$$

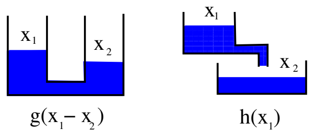


Figure 1. g–type (left) and h–type (right) flows in a fluid system.

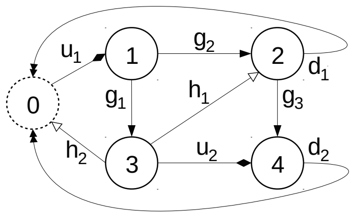


Figure 2. Graph of a network with natural dynamics (Example 1).

where $[\tilde{R}]_{ij} = \min\{R_{ij}, 0\}$. Matrix $[S\ R\ B]$ is the overall *incidence matrix* of the graph representing the network.

In the network graph, we distinguish between *g–type*, *h–type*, *u–type* and *d–type* arcs, associated respectively with the components of vector g^* (*i.e.*, with the columns of S), of vector h^* (with the columns of R), of the control vector u (with the columns of B) and of the demand vector d.

Definition 1 *If in a column of matrix R or of matrix B there is a single non–zero entry, we say that the corresponding link connects the network with the* external environment*, associated with the* external node *(*node 0*).*

In general, any arc connected with a single node of the graph represents a connection with the external environment. This is the case, for instance, of arcs associated with vector d.

Example 1 *In the network graph in Fig. 2 there are g–type flows:* $g^*(x) = [g_1(x_1 - x_3)\ \ g_2(x_1 - x_2)\ \ g_3(x_2 - x_4)]^\top$*; h–type flows:* $h^*(x) = [h_1(x_3)\ \ h_2(x_3)]^\top$*; controlled flows:* $u = [u_1\ \ u_2]^\top$ *and exogenous flows:* $d = [d_1\ \ d_2]^\top$*. The corresponding system has matrices*

$$S = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

We work under the following assumptions.

Assumption 1 *The control is componentwise bounded as*

$$u^- \le u \le u^+. \qquad (2)$$

Assumption 2 *Functions g_j^* and h_j^* are smooth and have positive derivative.*

Assumption 3 *There are no g–type flows from/to node* 0.[1]

[1] There is no point in considering g–type flows from/to node 0, since there are no state variables associated with it.

Definition 2 *A* path *on the graph is an oriented*[2] *sequence of distinct arcs, connecting two distinct nodes (and not including node* 0 *as an intermediate node). A path is* admissible *if: 1) it does not include d–type arcs; 2) whenever it includes h–type arcs, they have path–consistent orientation; other arcs can be included with arbitrary orientation.*[3]

In the graph in Fig. 2, for example, the path 4–3–2–1 is admissible, while 2–3–4 is not (because it includes h–type arc h_1, whose orientation is opposite to that of the path).

Definition 3 *The graph is* connected *if an oriented path exists connecting each pair of nodes (excluding node* 0*). The graph is* strongly connected *if an oriented* admissible *path exists connecting each pair of nodes (excluding node* 0*). The graph is* externally connected *if, for each node, an oriented admissible path exists leading to node* 0.

The graph in Fig. 2 is both strongly connected and externally connected. If we swap g–type and h–type arcs, we get a graph which is not strongly connected (node 1 cannot be reached from the other nodes), but is still externally connected (from each node, an admissible path leads to node 0). If, instead, in Fig. 2 we remove the arcs u_1 and h_2, the graph is still strongly connected, but not externally connected.

3 Stabilizability conditions

Consider the following definition.

Definition 4 *A feedback control is* network–decentralized *if any agent u_j decides its strategy based just on the state variables to which it is directly connected. Precisely, if p and q index the non–zero elements of column B_j, then $u_j = \Phi_j(x_q, x_p)$.*

In the example in Fig. 2, u_1 depends on x_1 only, while u_2 depends on x_3 and x_4 only.

Assumption 4 *The demand is constant, $d(t) = d$, and an equilibrium vector $\bar{x}$ exists corresponding to a control $\bar{u}$ that strictly satisfies* (2)*,* i.e.*, $u^- < \bar{u} < u^+$:*

$$0 = Sg^*(\bar{x}) + Rh^*(\bar{x}) + B\bar{u} + d.$$

Denoting by $v = u - \bar{u}$ and $z = x - \bar{x}$, without restrictions we can consider the stabilization problem for the shifted system

$$\dot{z} = Sg(z) + Rh(z) + Bv, \tag{3}$$

where g, h are the shifted functions $g(z) = g^*(z+\bar{x}) - g^*(\bar{x})$, $h(z) = h^*(z+\bar{x}) - h^*(\bar{x})$, such that $g(0) = 0$, $h(0) = 0$. Hence, *stability* is now referred to the nominal equilibrium $\bar{z} = \bar{v} = 0$. Accordingly, the constraints are

$$v^- \le v \le v^+, \tag{4}$$

where $v^- = u^- - \bar{u} < 0$ and $v^+ = u^+ - \bar{u} > 0$.

We assume that the equilibrium $(\bar{x}, \bar{u})$ is given and we do not consider the problem of regulating the steady–state value. For set–point regulation, the reader is referred for instance to Haddad, Hayakawa, & Bailey (2006); Lee & Ahn (2015) and the references therein.

Remark 1 *Assuming that $\bar{u}$ satisfies the constraints is crucial (Blanchini, Miani, & Ukovich 2000). Conversely, the requirement that d is constant can be removed. Moreover, due to parameter uncertainties, the equilibrium condition might not be known exactly, so that $Sg^*(\bar{x}) + Rh^*(\bar{x}) + B\bar{u} + d = \Delta \neq 0$ and equation* (3) *becomes $\dot{z} = Sg(z) + Rh(z) + Bv - \Delta$. These issues will be discussed in Section 5.*

Definition 5 *Given the vector bound $v^- \le v \le v^+$, the* saturation function *is componentwise defined as*

$$[sat(v)]_j = \begin{cases} v_j^+ & \text{if } v_j > v_j^+ \\ v_j & \text{if } v_j^- \le v_j \le v_j^+ \\ v_j^- & \text{if } v_j < v_j^- \end{cases} \tag{5}$$

We consider the following *saturated network–decentralized control* (Bauso, Blanchini, Giarré, & Pesenti 2013):

$$v = \mathrm{sat}(-\gamma B^\top z), \tag{6}$$

where γ is a positive gain (the higher γ, the stronger the control action).

A control similar to (6) is

$$v = \mathrm{sat}(-\gamma \tilde{B}^\top z), \tag{7}$$

where $\tilde{B} = \min\{B, 0\}$, componentwise. This control is suitable for applications in which the controlled flow in each link is decided based on the departure node only (*e.g.*, data communication networks), and will be considered in Remark 3.

We now state the main result of this section.

Theorem 1 *Under Assumptions 1–4, if the system graph is strongly connected, the following statements are equivalent.*

i) System (3) *can be globally uniformly stabilized*[4] *to $z = 0$.*
ii) Matrix $[S\ R\ B]$ has row rank n.
iii) The system graph is externally connected.
iv) System (3) *can be globally uniformly stabilized (to $z = 0$) by the network–decentralized control* (6).

Proof *iv)* $\Rightarrow$ *i)* is obvious.
i) $\Rightarrow$ *ii)*: if $\mathrm{rank}[S\ R\ B] < n$, then a left kernel exists. Consider

[2] Oriented means that the two paths from node i to node j and from node j to node i are different.

[3] Equivalently, we can replace each g–type and u–type arc by two arcs in opposite directions, and consider directed paths only.

[4] System $\dot{z}(t) = f(z(t), v(t))$, $f(0,0) = 0$, is globally uniformly asymptotically stabilizable if a control law $v(z)$ can be chosen so that: (a) $\forall \varepsilon$, $\exists \delta$ such that $\|z(0)\| \le \delta \Rightarrow \|z(t)\| \le \varepsilon\ \forall t \ge 0$; and (b) $\forall \mu > \varepsilon > 0$, $\exists T_{\mu,\varepsilon}$ such that $\|z(0)\| \le \mu \Rightarrow \|z(t)\| \le \varepsilon\ \forall t \ge T_{\mu,\varepsilon}$.

$\zeta \in \mathbb{R}^n$, $\zeta \neq 0$, such that $\zeta^\top [S\ R\ B] = 0$. We have

$$\frac{d}{dt}\zeta^\top z = \zeta^\top [Sg(z) + Rh(z) + Bv] = 0,$$

hence $\zeta^\top z(t) = \zeta^\top z(0) = \text{const}$. Then if $\zeta^\top z(0) \neq 0$, $z(t)$ cannot converge to 0.
ii) $\Rightarrow$ *iii)*: since the graph is strongly connected by assumption, $\mathrm{rank}[S\ R\ B] = n$ implies connection with the external node 0, *i.e.*, the existence of at least one column having a single non–zero element.

For instance, if in Example 1 (Fig. 2) we remove u_1 and h_2 (corresponding to columns with a single non–zero element), the system is no longer externally connected and

$$\mathrm{rank}[S\ R\ B] = \mathrm{rank}\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = 3 < 4.$$

Proving *iii)* $\Rightarrow$ *iv)* requires a Lemma.

Lemma 1 *Given functions $g(z)$ and $h(z)$ as in our assumptions, and matrices B, S and R, there exist positive definite diagonal matrix functions $D_v(z)$, $D_h(z)$ and $D_g(z)$ such that*

$$Bsat(-\gamma B^\top z) = -\gamma B D_v(z) B^\top z, \tag{8}$$

$$Sg(z) = -S D_g(z) S^\top z, \tag{9}$$

$$Rh(z) = -R D_h(z) \tilde{R}^\top z, \tag{10}$$

where $\tilde{R} = \min\{R, 0\}$, componentwise. Moreover, in any bounded neighborhood of $z = 0$, $\|z\| \leq \mu$, there exist two numbers $0 < \delta^- < \delta^+$ such that

$$\delta^- \leq\ [D_g(z)]_{ii},\ [D_h(z)]_{ii},\ [D_v(z)]_{ii}\ \leq \delta^+, \quad \forall\, i. \tag{11}$$

Proof Equation (8) is a standard property of the saturation function: componentwise $\mathrm{sat}(v_i) = [D_v]_{ii} v_i$, where $[D_v]_{ii}$ is a suitable number (see for instance Blanchini & Miani 2015). To prove (9), first note that any strictly increasing function f with $f(0) = 0$ can be written as

$$f(\xi) = \int_0^\xi f'(\sigma) d\sigma = \left[\int_0^1 f'(\lambda\xi) d\lambda\right]\xi,$$

where we have changed variable $\xi\lambda = \sigma$. Consider the generic column S_j of S and the corresponding term g_j. As we have seen, g_j is of the form $g_j = g_j(-S_j^\top z)$ and then

$$S_j g_j(-S_j^\top z) = -S_j \underbrace{\left[\int_0^1 g_j'(\lambda S_j^\top z) d\lambda\right]}_{\doteq D_{jj}(z)} S_j^\top z.$$

$D_{jj}(z)$ is strictly positive (being the integral of a positive function on a non–zero interval), lower and upper bounded in any bounded domain. The proof for (10) is analogous. ■

Then, in view of Lemma 1, we can write system (3) with the control (6) as

$$\dot{z} = -[S D_g(z) S^\top + R D_h(z) \tilde{R}^\top + \gamma B D_v(z) B^\top] z =$$

$$\begin{bmatrix} S & R & B \end{bmatrix} \begin{bmatrix} -D_g(z) & 0 & 0 \\ 0 & -D_h(z) & 0 \\ 0 & 0 & -\gamma D_v(z) \end{bmatrix} \begin{bmatrix} S^\top \\ \tilde{R}^\top \\ B^\top \end{bmatrix} z \doteq A(D) z.$$

In Example 1, with $D = -\mathrm{diag}(D_1, D_2, \ldots, D_7)$ and $\gamma = 1$, we obtain $A(D) =$

$$\begin{bmatrix} -(D_1+D_2+D_6) & D_2 & D_1 & 0 \\ D_2 & -(D_2+D_3) & D_4 & D_3 \\ D_1 & 0 & -(D_1+D_4+D_5+D_7) & D_7 \\ 0 & D_3 & D_7 & -(D_3+D_7) \end{bmatrix}.$$

Note that $A(D)$ can be non–symmetric due to h–type arcs.

We can now prove *iii)* $\Rightarrow$ *iv)*. Matrix $A(D)$, which has strictly negative diagonal entries and non–negative off–diagonal entries, is column diagonally–dominant:

$$-[A(D)]_{jj} \geq \sum_{i=1,\ i\neq j}^{n} [A(D)]_{ij}.$$

Indeed $A(D)$ is the sum of rank–one matrices of the form $-S_j[D_g]_{jj}S_j^\top$, $-R_j[D_h]_{jj}\tilde{R}_j^\top$ or $-\gamma B_j[D_v]_{jj}B_j^\top$, which we name *components*. Each component has at most four non–zero elements, all of equal magnitude ($[D_g]_{jj}$, $[D_h]_{jj}$ or $\gamma[D_v]_{jj}$), and is (at least weakly) diagonally dominant. There are three types of components.

- Components of the form $-\gamma B_j[D_v]_{jj}B_j^\top$ (or $-S_j[D_g]_{jj}S_j^\top$), where B_j (or S_j) connects two nodes. Two of the non–zero entries, negative, are on the diagonal; the other two, positive, are on the corresponding two columns and rows.
- Components of the form $-R_j[D_h]_{jj}\tilde{R}_j^\top$, where R_j connects two nodes. There are two non–zero entries, of opposite sign; the negative one is on the diagonal, the positive one on the same column.
- Components of the form $-\gamma B_j[D_v]_{jj}B_j^\top$ (or $-R_j[D_h]_{jj}\tilde{R}_j^\top$), where B_j (or R_j) connects with node 0. There is a single negative diagonal entry.

The fact that the system can be written as $\dot{z} = A(D(z))z$, with $A(D(z))$ diagonally dominant having negative diagonal entries, implies that the function $V(z) = \|z\|_1$, the 1–norm, is a (weak) Lyapunov function (Willems 1976; Maeda, Kodama, & Ohta 1978; Blanchini & Miani 2015). This proves Lyapunov stability, but unfortunately not asymptotic stability.
Since our graph is strongly connected, matrix $A(D(z))$ is irreducible: no ordering of the variables exists such that

$A(D(z))$ assumes a block–triangular form:

$$A(D(z)) = \begin{bmatrix} A_{11}(D(z)) & \mathbf{0} \\ A_{21}(D(z)) & A_{22}(D(z)) \end{bmatrix}. \qquad (12)$$

By contradiction, assume that the system can be brought in the block–triangular form (12), with $A_{11}(D(z)) \in \mathbb{R}^{r\times r}$ and $A_{22}(D(z)) \in \mathbb{R}^{(n-r)\times(n-r)}$. Clearly, this would imply that there is no directed path from *any* of the nodes $r+1,\ldots,n$ to *any* node among $1,\ldots,r$, thus contradicting the assumption of strong connectivity. In view of the assumption of external connection, matrix $[S\ R\ B]$ has at least one column with a single non–zero element, hence there will be at least one component with a single (negative) diagonal entry.
Since $A(D(z))$ is irreducible, column diagonally dominant with at least one strictly dominant diagonal entry, and has negative diagonal entries, the 1–norm is a strong Lyapunov function and $\bar{z}=0$ is a globally asymptotically stable equilibrium (Willems 1976; Maeda, Kodama, & Ohta 1978; for further details, see Theorems 1 and 2 and the following remarks by Willems (1976) and Theorem 4.60 by Blanchini & Miani (2015), where a thorough proof is provided). ■

To ensure more flexibility in the control design, different gains can be chosen for different control components:

$$v = \mathrm{sat}(-\Gamma B^\top z), \qquad (13)$$

with $\Gamma = \mathrm{diag}(\gamma_1,\ldots,\gamma_m)$ a positive definite diagonal matrix.

Corollary 1 *Theorem 1 equivalently holds if the control* (6) *is replaced by the control* (13).

Proof From Lemma 1, $B\mathrm{sat}(-\Gamma B^\top z) = -(BD_v\Gamma B^\top z) = -(B\tilde{D}_v B^\top z)$, where $\tilde{D}_v$ is still a positive definite diagonal matrix. Hence, all the derivations can be carried out as in the case of a scalar γ. ■

If the graph is not strongly connected, the theorem holds in a weaker form: in general, stabilizability and network–decentralized stabilizability are not equivalent.

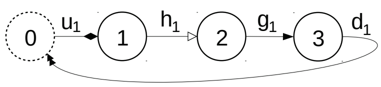


Figure 3. The graph in Example 2.

Example 2 *Consider the graph in Fig. 3, which is connected ($[S\ R\ B]$ has row rank $n=3$), but not strongly connected (there is no admissible path leading from nodes 2 and 3 to node 1). Hence the assumptions of Theorem 1 are not satisfied. If we assume a linear model $\dot{z} = Fz + Bv$, we get*

$$F(D) = \begin{bmatrix} -D_2 & 0 & 0 \\ D_2 & -D_1 & D_1 \\ 0 & D_1 & -D_1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

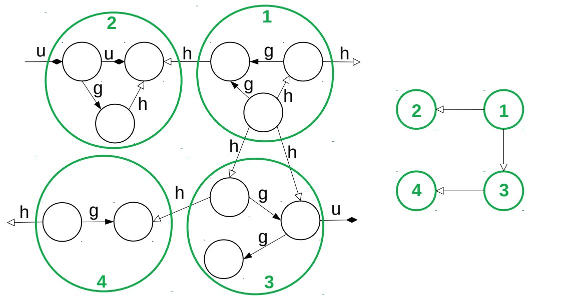


Figure 4. A graph where maximal strongly–connected clusters are encircled (left) and the corresponding aggregate graph (right).

The results in Theorem 1 cannot be applied to this case. Indeed, although the graph is not externally connected, the system is stabilizable (in fact it is reachable), but not in a decentralized way. A decentralized choice of v_1 should be a function of z_1 only, but no stabilization would be possible, since the system is undetectable from output $y = z_1$; hence, agent v_1 needs information also from z_2 and z_3.

We now relax the assumption of strong connectivity.

Theorem 2 *Under Assumptions 1–4, consider a system of the form* (3)*, whose graph is not strongly connected. Then the system can be globally uniformly stabilized (to $z=0$) by the network–decentralized control* (6) *if and only if the system graph is externally connected.*

Proof *Sufficiency*. If the system is not strongly connected, then we can consider *maximal strongly connected clusters of nodes* $\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_N$, such that: (a) for each pair of nodes $i,j \in \mathscr{C}_k$, admissible paths exist in both directions; (b) for each pair of nodes with $i \in \mathscr{C}_k$ and $j \notin \mathscr{C}_k$, an admissible path in at least one direction is missing (see Fig. 4). Consider the aggregate oriented graph formed by the clusters, as in Fig. 4: a directed arc connects cluster l to cluster k if an admissible path leads from a node in l to a node in k. This aggregate graph is acyclic. Indeed, the presence of a directed cycle, say, $\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_N,\mathscr{C}_1$, would imply that $\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_N$ form a strongly connected cluster, in contradiction with the fact that $\mathscr{C}_i$ are maximal.
The clusters can be associated with diagonal blocks of $A(D)$. Since the aggregate graph is acyclic, if we consider the control (6) as before, these blocks can be arranged so that $A(D)$ has a lower triangular form. In the case of Fig. 4 we have

$$A(D(z)) = \begin{bmatrix} A_{11}(D) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{21}(D) & A_{22}(D) & \mathbf{0} & \mathbf{0} \\ A_{31}(D) & \mathbf{0} & A_{33}(D) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{43}(D) & A_{44}(D) \end{bmatrix}.$$

All the diagonal blocks $A_{ii}(D)$ are irreducible matrices, because they correspond to strongly connected clusters, and weakly diagonally dominant. Moreover, the matrices $A_{ii}(D)$, $i = 1,2,\ldots N-1$, must have a diagonal entry that is strictly diagonally dominant, referred to $A_{ii}(D)$. Indeed there is a

connection either with another block (*i.e.*, there is a non–zero entry in a non–diagonal block, hence the corresponding diagonal element in the diagonal block is dominant), or with the external node; otherwise we would not have external connection. Finally, $A_{NN}(D)$ must have a strictly dominant diagonal entry due to the assumed connection with the external node. Then all the $A_{ii}(D)$ are irreducibly diagonally dominant and, since their diagonal entries are negative, all the corresponding subsystems are asymptotically stable. In view of the block–triangular form, this implies asymptotic stability of the overall system (Blanchini & Miani 2015).
Necessity. Assume that the system is not externally connected: nodes $1,2,\ldots,r$, forming the subset $\mathscr{C}$, are connected to node 0 (*i.e.*, a path leading to 0 starts from each of them), while nodes $r+1,r+2,\ldots,n$, forming the subset $\mathscr{D}$, are not connected to 0 (*i.e.*, no path starting from them leads to 0). Then $[S\ R\ B]$ can be partitioned as

$$[S \mid R \mid B] = \left[\begin{array}{cc|cc|cc} S_1 & \mathbf{0} & R_{11} & \mathbf{0} & B_1 & \mathbf{0} \\ \mathbf{0} & S_2 & R_{21} & R_{22} & \mathbf{0} & B_2 \end{array}\right],$$

with R_{21} non–negative. The zero blocks in S and B are due to the absence of u–type or g–type arcs (which can be crossed in both directions) connecting a pair of nodes belonging one to $\mathscr{C}$ and one to $\mathscr{D}$. The structure of R is motivated by the fact that no h–type arc can start from a node in $\mathscr{D}$ and reach a node in $\mathscr{C}$. Since no node in $\mathscr{D}$ is connected with node 0, all the columns of the sub–matrices S_2, R_{22} and B_2 have two non–zero elements, equal to -1 and 1. In particular, the sum of the entries in each column is 0.
Now we partition the control in two vectors, v_1 and v_2, corresponding to B_1 and B_2. If the control is decentralized, v_1 can be a function of z_1 only and v_2 of z_2 only: $v_1 = v_1(z_1)$, $v_2 = v_2(z_2)$. The overall system can be written as

$$\begin{aligned} \dot{z}_1 &= S_1 g_1(z_1) + R_{11} h_1(z_1) + B_1 v_1(z_1) \\ \dot{z}_2 &= S_2 g_2(z_2) + R_{21} h_1(z_1) + R_{22} h_2(z_2) + B_2 v_2(z_2), \end{aligned}$$

where $g_1(z_1)$ and $h_1(z_1)$ depend on z_1 only, while $g_2(z_2)$ and $h_2(z_2)$ on z_2 only.
Assume by contradiction that the closed–loop system is stable. If $z_1(0) = 0$, we would have $z_1(t) = 0$ and $v_1(t) = 0$ $\forall\ t \geq 0$. The second equation would become

$$\dot{z}_2 = S_2 g_2(z_2) + R_{22} h_2(z_2) + B_2 v_2(z_2).$$

Consider the function $\mathrm{sum}(z_2) = \bar{1}^\top z_2$; its derivative is $\bar{1}^\top \dot{z}_2 = \bar{1}^\top [S_2 g_2 + R_{22} h_2 + B_2 v_2] = 0$, because S_2, R_{22} and B_2 have zero–sum columns. Then, if $\mathrm{sum}(z_2(0)) = \kappa \neq 0$, $\mathrm{sum}(z_2(t)) = \kappa \neq 0\ \forall\ t \geq 0$, hence $z_2(t)$ does not converge to 0. ■

Remark 2 *The control in the original variables has the form $u = \bar{u} + sat[-\gamma B^\top (x - \bar{x})]$. The control agents must know the* local *equilibrium values of $\bar{u}$ and $\bar{x}$ associated with the arcs they control and with the nodes to which they are directly connected. The control $u = sat(-\gamma B^\top x)$, considered by Bauso, Blanchini, Giarré, & Pesenti (2013), may not be stabilizing (see Example 3 in Section 5).*

In the absence of a control action, we recover the stability result provided by Maeda, Kodama, & Ohta (1978); Jacquez & Simon (1993) for compartmental systems of the form $\dot{z} = Sg(z) + Rh(z)$. For these systems, the condition $\mathrm{rank}[S\ R] = n$ is equivalent to the existence of a "path to the outside world" according to Theorem 7 by Maeda, Kodama, & Ohta (1978).

Corollary 2 *If $B = 0$, then the system* (3) *is asymptotically stable (in $z = 0$) if and only if rank$[S\ R] = n$.*

Remark 3 *In the case of control* (7)*, the theory is essentially unchanged, apart from Definition 2 that must be revised: a path can be considered* admissible *only if both u–type and h–type arcs, whenever included, have path–consistent orientation. Then, along the same lines, it can be proved that* (7) *is stabilizing if and only if the overall system is externally connected. We omit the details for brevity.*

4 Asymptotic flow optimality via decentralized control

In this section we propose a network–decentralized strategy, inspired by (6), that considers just the marginally stable part of the system and achieves optimality at steady state for any d compatible with the flow constraints (as in Assumption 4). Given a system of the form (1), we assume that the nodes are grouped into *macro–nodes*. Each macro–node is a subsystem with compartmental dynamics:

$$\dot{x}_i = S_i g_i^*(x_i) + R_i h_i^*(x_i) + \sum_{j \in \mathscr{C}_i} B_{ij} u_j + d_i, \tag{14}$$

$i = 1,2,\ldots,N$, where $x_i(t) \in \mathbb{R}^{n_i}$ and $\mathscr{C}_i$ is the set that indexes control agents $u_j \in \mathbb{R}^{m_i}$ affecting macro–node i. Let $[S\ R] = \left[[S_1\ R_1]^\top\ [S_2\ R_2]^\top \ldots [S_N\ R_N]^\top\right]^\top$. This model accounts for the case in which there is no shared dynamics between the macro–nodes, except that pairs of them can be influenced by the same control agent u_i. It is relevant, for instance, to model traffic between nodes, in each of which the traffic splits in several direction according to some dynamic model (Blanchini, Giordano, & Montessoro 2014; see Fig. 5, illustrating the example in Section 6).

Assumption 5 *The uncontrolled system* (1)*, composed of subsystems of the form* (14) *with $u_j = 0$, is* input–to–state stable *within the left kernel of* $[S\ R]$*: for each perturbation $\bar{d}$ that is orthogonal to the left kernel of* $[S\ R]$ *(*i.e.*, $E^\top [S\ R] = 0$ implies $E^\top \bar{d} = 0$), there exist a unique steady state $\bar{x}$ such that $0 = Sg^*(\bar{x}) + Rh^*(\bar{x}) + \bar{d}$. Moreover, for all $\delta \in ker[S\ R]^\top$, $\|\delta\| \leq \bar{\delta}$, such that $d(t) = \bar{d} + \delta$, and all initial conditions $x(0) = \bar{x} + z$, with $z \in ker[S\ R]^\top$, we have*

$$\|x(t) - \bar{x}\| \leq C_1 \bar{\delta} + C_2 \phi(t) \|x(0) - \bar{x}\|,$$

where C_1 and C_2 are positive constants, while $\phi(t)$ is a continuous positive function, strictly decreasing and converging to 0 as $t \to \infty$.

For linear systems, this is equivalent to requiring that A is asymptotically stable, or marginally stable without purely imaginary eigenvalues (cf. Definition 7 in Section 4.1).

Here we are reconsidering the original variable x; we assume that 0 is the *reference level*, which is not necessarily the steady state $\bar{x}$. Note that, being the demand d unknown to the controller, we cannot assure exact convergence to the desired value.

Lemma 2 *Under Assumptions 1–5, denote by $E^\top$ a basis of the left kernel of $[S\ R]$: $E^\top[S\ R]=0$. Then the control*

$$u = sat(-\gamma B^\top E E^\top x) \tag{15}$$

assures that from any initial condition the system converges to a finite ball $\|x\| \le \mu$, for some $\mu > 0$, if and only if

$$u^- < -E^\top d < u^+. \tag{16}$$

Proof Applying the transformation $e = E^\top x$ and $f = F^\top x$, with $[E\ F]$ invertible, the system can be represented as

$$\dot{e} = E^\top B u + E^\top d \tag{17}$$

$$\dot{f} = F^\top S\hat{g}(e,f) + F^\top R\hat{h}(e,f) + F^\top B u + F^\top d \tag{18}$$

The subsystem (17) can be stabilized iff $u^- < -E^\top d < u^+$ (Blanchini, Miani, & Ukovich 2000) and the control $u = \mathrm{sat}(-\gamma B^\top E e)$ assures convergence of e to some $\bar{e}$. Equation (18) asymptotically becomes

$$\dot{f} = F^\top S\hat{g}(\bar{e},f) + F^\top R\hat{h}(\bar{e},f) + F^\top \delta(t),$$

where the perturbation $\delta = [S(\hat{g}(e,f)-\hat{g}(\bar{e},f)) + R(\hat{h}(e,f)-\hat{h}(\bar{e},f)) + B\bar{u} + d]$ is bounded and directed along the kernel of the transformed matrix $[\hat{S}\ \hat{R}] = F^\top[S\ R]$. Then, in view of Assumption 5, $f(t)$ converges to a bounded equilibrium. ■

The set of all equilibrium conditions $\bar{x}$ and $\bar{u}$ is parameterized by the equation $Sg(\bar{x}) + Rh(\bar{x}) + B\bar{u} + d = 0$, or

$$S_i g_i(\bar{x}_i) + R_i h_i(\bar{x}_i) + \sum_{j\in\mathscr{C}_i} B_{ij}\bar{u}_j + d_i = 0, \quad \forall\ i. \tag{19}$$

Based on Lemma 2 and the decomposition (17)–(18), the following result holds.

Lemma 3 *Let d and $\bar{x}$ be as in Assumption 5. Then, $\bar{u}$ is an admissible steady–state control input if and only if*

$$\bar{u} \in \Omega(d) = \{u \in \mathscr{U} :\ E^\top B u + E^\top d = 0\}. \tag{20}$$

The next property from Bauso, Blanchini, Giarré, & Pesenti (2013) is valid for pure buffer systems ($\dot{x} = Bu + d$).

Theorem 3 *Assume $E = I$ (the identity), $S = 0$, $R = 0$ and the set $\Omega(d)$ in* (20) *has a non–empty interior. Then the (network–decentralized) control* (15) *assures convergence to some equilibrium $\bar{x}$ and the corresponding control value at steady state has minimum Euclidean norm:*

$$\lim_{t\to\infty} u(t) = \bar{u} = \arg\min_{u\in\Omega(d)} \|u\|. \tag{21}$$

We now show that the asymptotic minimum–norm property holds as well in the case of compartmental systems.

Theorem 4 *If condition* (16) *of Lemma 2 is satisfied and Assumptions 1–5 hold, then the control* (15) *is asymptotically optimal,* i.e.*, converges to the vector in $\Omega(d)$ having minimum Euclidean norm in the sense of* (21)*.*

Proof We have $u(t) \to \bar{u}$, where $\bar{u}$ is a finite value, because the control is stabilizing and continuous. Indeed, from (17)–(18), we have that $e(t)$ converges to some finite $\bar{e}$ and, by assumption, $f(t)$ converges to some finite $\bar{f}$.
The control u is a function of $e = E^\top x$ only:

$$u = \mathrm{sat}(-\gamma B^\top E E^\top x) = \mathrm{sat}(-\gamma B^\top E e).$$

Consider equation (17) to get

$$\dot{e} = E^\top B \mathrm{sat}(-\gamma B^\top E e) + E^\top d.$$

In view of Theorem 3 (considering $\tilde{B} = E^\top B$ and $\tilde{d} = E^\top d$), the control u converges to the minimum norm control $\bar{u}$ inside $\Omega(d)$. ■

A suitable choice of matrix E allows us to achieve asymptotic optimality in a network–decentralized way, according to the following definition that extends Definition 4 to the macro–node case.

Definition 6 *Denote by $\mathscr{S}_j$ the set that indexes macro–nodes directly affected by agent u_j,* i.e.*, subsystems associated with the non–zero components of the block–column B_j. The control is* network–decentralized *if any agent $u_j \in \mathbb{R}^{m_i}$ decides its action based on the state variables in $\mathscr{S}_j$ only: $u_j = \Phi_j(x_k,\ k \in \mathscr{S}_j)$.*

For instance, if $B_j = [\ 0\ \ B_{2j}^\top\ \ 0\ \ B_{4j}^\top\ \ B_{5j}^\top\ ]^\top$, then u_j is a function of vectors x_2, x_4, and x_5. The control (15) is network–decentralized if matrix E is chosen of a proper block–diagonal form.

Proposition 1 *Given the system composed by decoupled subsystems of the form* (14)*, take matrix E as*

$$E^\top = \mathrm{blockdiag}\{E_1^\top, E_2^\top, \dots, E_N^\top\}, \tag{22}$$

where $E_i^\top$ is a left kernel of S_i: $E_i^\top S_i = 0$. Then the control (15) *is network–decentralized.*

Remark 4 *The proposed control choice minimizes the* controlled flow *(not the overall flow). Optimality with respect to the weighted norm $u^\top \Sigma^2 u$, with diagonal Σ, can be easily achieved by scaling the columns of matrix B as $B\Sigma^{-1}$. Achieving a minimum* overall flow *is not a simple task in our* structural *setup, since the functions g and h are not known.*

4.1 The linear case

When g and h are linear, (14) becomes

$$\dot{x}_i = A_i x_i + \sum_{j \in \mathscr{C}_i} B_{ij} u_j + d_i, \tag{23}$$

where A_i are Metzler matrices ($[A_i]_{pq} \geq 0$ for $p \neq q$).

Definition 7 *System* (23) *is a* buffer system *if* $A_i = 0$. *It is an* extended buffer system *if A_i is either asymptotically stable or marginally stable with marginally stable eigenvalue $\lambda = 0$ (of any multiplicity).*

Proposition 2 *An extended buffer system of the form* (23) *is stable in the left kernel,* i.e.*, satisfies Assumption 5.*

Proof Any basis $E^\top$ of the left kernel of A can be written as in (22), where $E_i^\top$ is a left kernel of A_i ($E_i^\top A_i = 0$). Take a complementary basis $F_i^\top$ such that $[E\ F]$ is invertible, and consider the transformation

$$T = \text{blockdiag}\left\{ \begin{bmatrix} E_1^\top \\ F_1^\top \end{bmatrix}, \begin{bmatrix} E_2^\top \\ F_2^\top \end{bmatrix}, \ldots, \begin{bmatrix} E_N^\top \\ F_N^\top \end{bmatrix} \right\}.$$

The transformed matrix $\hat{A} = T^{-1}AT$ is block–diagonal and each of its blocks has zero sub–blocks of the same dimension of the left kernel of A_i (since the system is stable by assumption, the eigenvalue 0 does not have Jordan blocks of dimension greater that 1):

$$\hat{A} = \text{blockdiag}\left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ P_1 & Q_1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ P_2 & Q_2 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ P_N & Q_N \end{bmatrix} \right\},$$

where matrices Q_i have no zero eigenvalues, hence are asymptotically stable. If d is orthogonal to the left kernel of A, then the transformed $\hat{d}$ has zero components corresponding to the zero blocks of $\hat{A}_i$ and the system $\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{d}$ is the parallel of systems of the form

$$\frac{d}{dt}\hat{x}_i(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ P_i & Q_i \end{bmatrix} \hat{x}_i(t) + \begin{bmatrix} \mathbf{0} \\ \hat{d}_i \end{bmatrix},$$

which satisfy Assumption 5 (because their state is bounded). Hence the overall system satisfies the assumption. ■

The following proposition is a special case of Theorem 1.

Proposition 3 *An extended buffer system is stabilizable iff rank*$[A \quad B] = n$.

If A_i are continuous–time Markov matrices, a candidate network–decentralized control is

$$u(t) = \text{sat}[-\gamma B^\top H x(t)], \tag{24}$$

with $\gamma > 0$ and, denoting by $e_n = \left[\frac{1}{\sqrt{n}}\ \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}} \right]^\top$,

$$H = \text{blockdiag}\{e_{n_1}e_{n_1}^\top, \ldots, e_{n_i}e_{n_i}^\top, \ldots, e_{n_N}e_{n_N}^\top\}. \tag{25}$$

This strategy can stabilize the system *robustly*, *i.e.*, without any knowledge about the Markov chain parameters (Blanchini, Giordano, & Montessoro 2014).

Theorem 5 *Assume that the dominant eigenvalue $\lambda = 0$ is simple for all matrices A_i.*[5] *Then the network–decentralized control* (24)–(25) *robustly stabilizes the system and is asymptotically optimal in norm.*

5 Robustness and positivity constraints: some remarks

The proposed control is intrinsically robust, being functions g^* and h^* unknown. It is robust even under switching topologies, since we have absorbed the closed loop system in a linear differential inclusion $\dot{z} = A(D)z$. If matrices S, R, B are switching inside a set $\{S_k, R_k, B_k\}$, and the conditions of Theorems 1 or 2 are satisfied for each k, we have a set of linear differential inclusions $\dot{z} = A_k(D)z$, all sharing the 1–norm as a Lyapunov function, and asymptotic stability of the closed loop system is preserved (Blanchini & Miani 2015). As mentioned earlier, the unknown exogenous demand d can be time varying, possibly due to uncertainties on the equilibrium condition. This is not an issue in our setup. In fact, assume that a perturbation is present; then

$$\dot{z}(t) = A(D(t))z(t) + \Delta(t).$$

Clearly, exact convergence to 0 cannot be assured. However, if $\|\Delta(t)\| \leq \Delta_{max}$, a robust asymptotic bound of the form

$$\limsup_{t \to \infty} \|z(t)\| \leq Z$$

is guaranteed if the linear differential inclusion is stable. The size of Z depends on Δ_{max} and the specific parameters. Therefore, stability/boundedness can be assured only if the value of Δ_{max} is compatible with the control constraints.

In some applications, positivity of the variables is required. Since compartmental systems are positive (*i.e.*, the positive orthant is positively invariant), a legitimate question is whether the control action preserves positivity.
Consider for brevity the linear case, in which A_i are Metzler matrices, and the control $u = \text{sat}(-\gamma B^\top x)$. Since the term $B\text{sat}(-\gamma B^\top x)$ can be written as $[-\gamma B D_v(x) B^\top]x$, for some state–dependent positive diagonal $D_v(x)$, we have

$$\dot{x} = Ax - \gamma B D_v(x) B^\top x + d.$$

If B is an incidence matrix, $[-\gamma B D_v(x) B^\top]$ is Metzler. Hence, if d is a positive vector, the overall system is positive.
Conversely, the control (15), $u = \text{sat}(-\gamma B^\top E E^\top x)$, destroys the positivity of the system. In practical applications such as flow systems, if a “buffer” becomes empty the control must be inhibited if it tries to force an outgoing flow (this might introduce chattering). A typical solution is to impose a non–zero reference level, greater than the physical zero level.

[5] A sufficient (but not necessary) condition is that A_i is irreducible.

Unlike (6) ($u = \bar{u} + \text{sat}[-\gamma B^\top(x - \bar{x})]$) and (15), the control $u = \text{sat}(-\gamma B^\top x)$ by Bauso, Blanchini, Giarré, & Pesenti (2013), asymptotically optimal if $A = 0$, is not optimal when $A \neq 0$ and might even lead to instability, even if the demand flow is compatible with the constraints.

Example 3 *Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

with $0 \leq u_i \leq 1$, $d_1 = 0.8$ and $d_2 = 1$. This demand is compatible with the constraints. The saturated control is $u_i = sat(\gamma x_i)$. The variable x_2 converges to the steady state $\bar{x}_2 = \frac{1}{1+\gamma} d_2$, with $d_2 = 1$; the outgoing flow is $\bar{u}_2 = \frac{\gamma}{\gamma+1} < 1$, so there is no saturation. The equation for x_1 is then

$$\dot{x}_1 = d_1 + \frac{1}{1+\gamma} d_2 - u_1 = 0.8 + \frac{1}{1+\gamma} - u_1.$$

Since $u_1 \leq 1$, the buffer x_1 diverges if γ is not large enough (it must be $\gamma > 4$). The control (24) *is instead*

$$u_1 = u_2 = sat\left[\frac{\gamma}{2}(x_1 + x_2)\right].$$

Considering the variable $y = x_1 + x_2$, we get

$$\dot{y} = d_1 + d_2 - u_1 - u_2 = 1.8 - 2sat\left(\frac{\gamma}{2} y\right),$$

Then $y \to 1.8/\gamma$ and the control converges to $\bar{u}_1 = \bar{u}_2 = 0.9$, which satisfies the constraints and has minimum norm. The control (6), *$u_i = sat[\gamma(x_i - \bar{x}_i)] + \bar{u}_i$, is stabilizing (although not optimal) for any admissible $\bar{x}$ and $\bar{u}$.*

6 Example

We consider data transmission systems (as in Fig. 5a) in which the macro–nodes are routers, internally modeled as a network with a central node, providing switching capabilities, and border nodes, representing the queues and the interfaces toward physical links. Data can be transmitted from a macro–node to another (see Fig. 5b), so that the buffer levels in each router vary due to three types of flows:

- the uncontrolled flow coming from the internal network and directed elsewhere;
- the controlled flow coming from other routers and directed to the internal network;
- the controlled transiting flow, coming from and directed to other routers.

Plain arrows represent controlled flows, while dashed arrows represent uncontrolled flows. The internal traffic in each macro–node splits in buffers with different destinations according to some probability distribution (see Fig. 5c): traffic splitting in each macro–node is modeled as a continuous–time Markov chain. Disk–headed arrows represent stochastic splitting. Consider for instance the macro–node A in Fig. 5c. All the traffic arriving at the central node, denoted by IA, splits in several directions, namely from IA to AB, AC, AD, and to AA, the buffer for the data directed into the local network A. However, in case of congestion of some link, the traffic directed to some node can be reconsidered, with a probability originated by an unknown re–routing criterion. This is represented by the arrows from AB, AC, AD to IA. The internal dynamics of the macro–node are modeled by a continuous–time Markov matrix:

$$A_A = \begin{bmatrix} -(\alpha_{AA} + \alpha_{AB} + \alpha_{AC} + \alpha_{AD}) & \alpha_{BA} & \alpha_{CA} & \alpha_{DA} & \alpha_{AA'} \\ \alpha_{AB} & -\alpha_{BA} & 0 & 0 & 0 \\ \alpha_{AC} & 0 & -\alpha_{CA} & 0 & 0 \\ \alpha_{AD} & 0 & 0 & -\alpha_{DA} & 0 \\ \alpha_{AA} & 0 & 0 & 0 & -\alpha_{AA'} \end{bmatrix}.$$

The first variable represents the arrival node; α_{AA}, α_{AB}, α_{AC}, α_{AD} are the probabilities that, in time dt, a packet is transferred to AA, AB, AC, AD. Conversely, α_{BA}, α_{CA}, α_{DA}, $\alpha_{AA'}$ are the probabilities that, in time dt, a packet is sent back to IA, from AB, AC, AD or AA, for re–routing. The matrices for the other macro–nodes are determined likewise. Matrix B is an incidence matrix: each column B_j corresponds to a controlled arc connecting two macro–nodes, or leaving a macro–node. The columns are determined as follows.

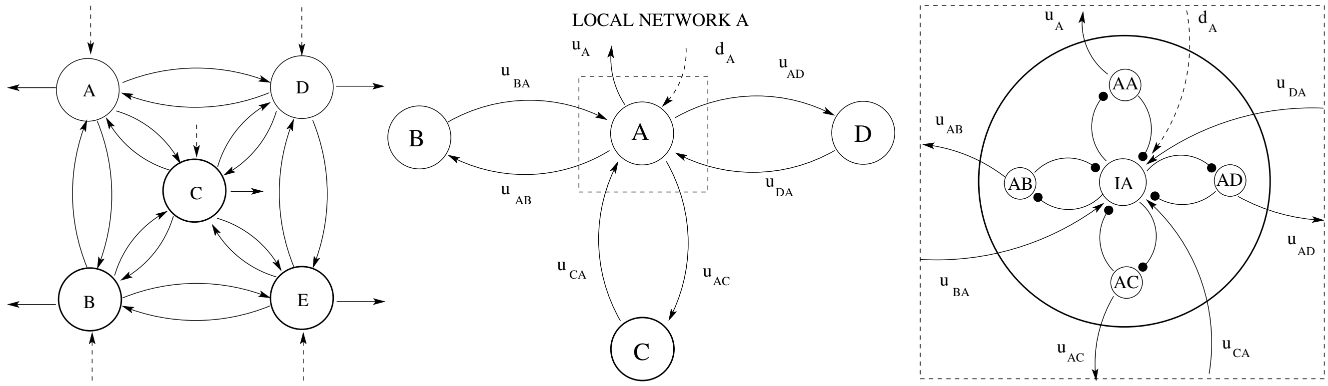
- If an arc connects a macro–node to the local network, there is a -1 in the row corresponding to the exit node of the macro–node, directing the traffic to the local network (node AA in Fig. 5c).
- If an arc connects two macro–nodes, say A to B , there is a -1 in the row corresponding to the node AB of macro–node A (directing the traffic to B) and a 1 in the arrival node IB of macro–node B.

At each node, the probability distribution is *unknown* and thus cannot be used for control purpose. Given the transmission network in Fig. 5a, we compare the behavior of the network when three different control strategies are applied:

- the saturated control $u = \text{sat}(-\gamma B^\top x)$ in (6);
- the H–saturated control $u = \text{sat}(-\gamma B^\top H x)$ in (24)–(25);
- the control $u = \text{sat}(-\gamma \tilde{B}^\top x)$ in (7).

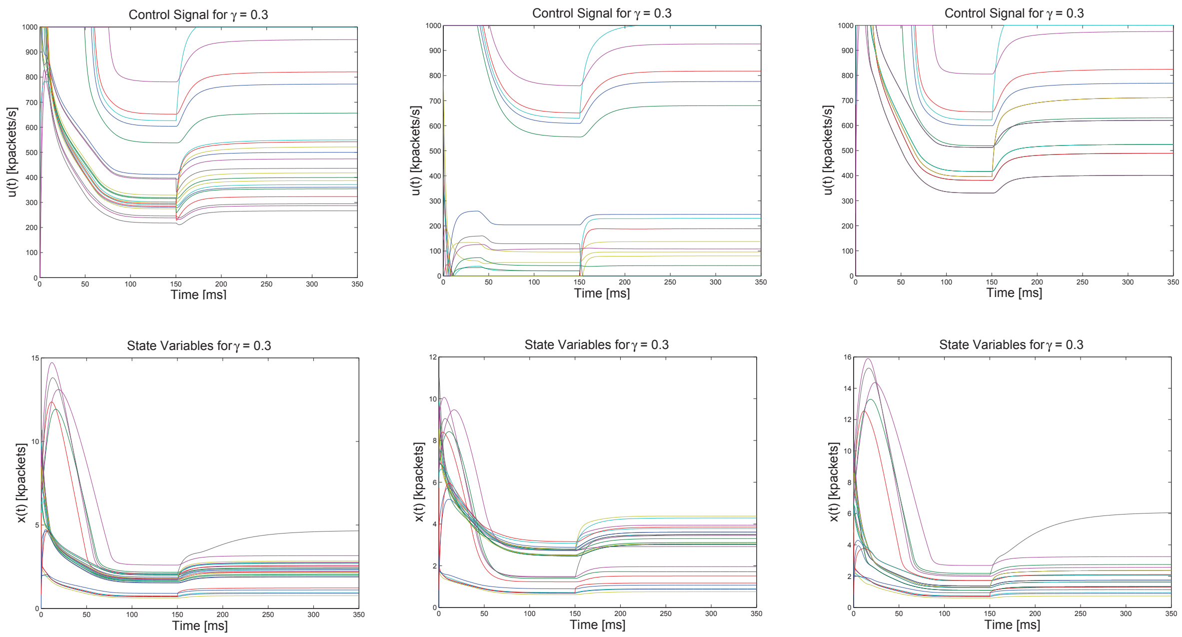
The control components were bounded in the interval $[0 \;\; 1]$ Mpackets/s. In the simulations shown in Fig. 6, the component of d affecting node D was suddenly increased to three times its initial value, to represent the case in which node D suddenly increases its traffic in all directions. In Fig. 6 we see that, with a suitable choice of γ, the H–saturated control guarantees the fastest convergence and the shortest queues in the buffers. Buffer queues are important, being related to delays in the network. The asymptotic value of the control is optimal, as expected: $u = [0.776 \; 0.68 \; 0.817 \; 1 \; 0.926 \; 0.096 \; 0 \; 0 \; 0.041 \; 0 \; 0.230 \; 0 \; 0.137 \; 0 \; 0.246 \; 0 \; 0.189 \; 0 \; 0.108 \; 0.081 \; 0]^\top$ is the minimum norm control (we checked it via CVX).

If γ is taken too large, the H–saturated control can no longer ensure positivity of the state variables. To have non–negative buffer levels, we need to stop the flow (forcing



(a) A 5–macro–nodes network. (b) Data flows among routers (macro–nodes). (c) Splitting dynamics in macro–node A.

Figure 5. A communication network (a), in which packets flow among routers (b), each seen as a macro–node having splitting dynamics (c).



(a) $u = \mathrm{sat}(-\gamma B^\top x)$ (b) $u = \mathrm{sat}(-\gamma B^\top H x)$ (c) $u = \mathrm{sat}(-\gamma \tilde{B}^\top x)$

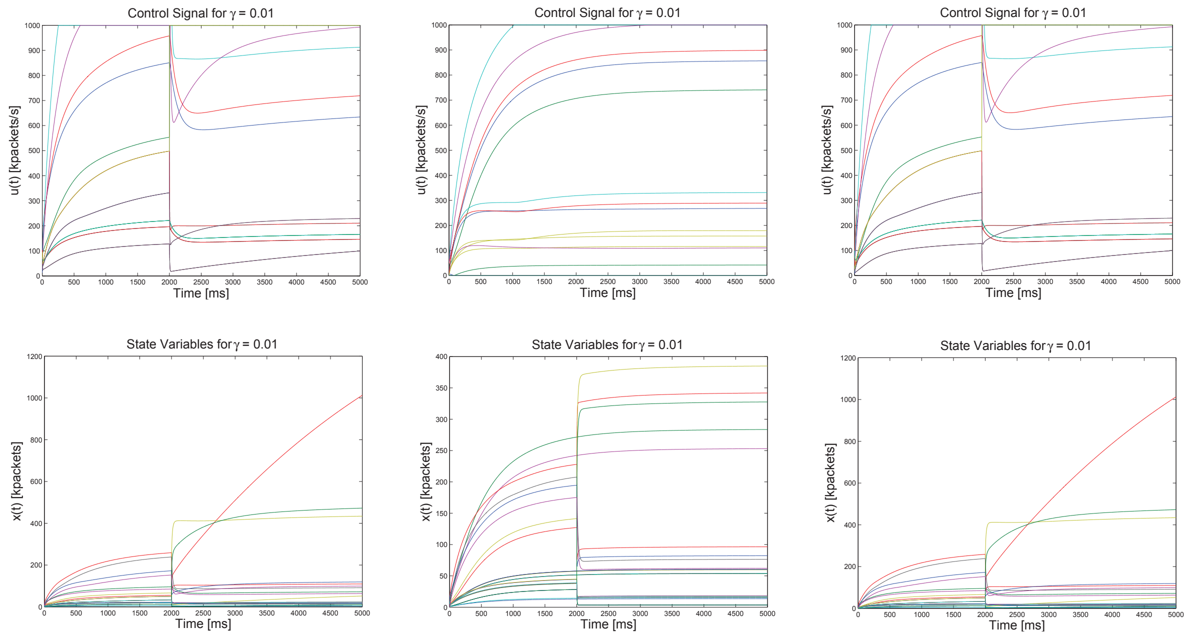
Figure 6. Simulations of the network in Fig. 5a. Given $\gamma = 0.3$ and common random initial conditions, the three considered control strategies are compared: the evolution of the control action u is in the above row, of the buffer levels x in the bottom row. Parameter values for macro–node A were $\alpha_{AA} = \alpha_{AB} = \alpha_{AC} = \alpha_{AD} = 1$, $\alpha_{BA} = \alpha_{CA} = \alpha_{DA} = 0.25$, $\alpha_{AA'} = 0.05$, and analogously for the other macro–nodes. The value of d was initially $[0.6\ \ 0.2\ \ 0.7\ \ 0.5\ \ 1.2]^\top$, then at $t = 150$ it was switched to $d = [0.6\ \ 0.2\ \ 0.7\ \ 1.5\ \ 1.2]^\top$.

$u = 0$) associated with arcs coming out from empty buffers; this discontinuity in the control may cause chattering.

In the simulation in Fig. 7, the Markov parameters are suddenly changed: $\alpha_{DE}, \alpha_{EB}, \alpha_{BB}$ are switched from 0.05 to 1, modeling the case in which node D increases the traffic to node E and node E the traffic to node B, resulting in a large traffic through nodes D–E–B. Again, the H–saturated control ensures a much faster convergence, lower buffer levels and smoother transitions. The resulting asymptotic value $u = [0.8579\ 0.742\ 0.9\ 1\ 1\ 0.1158\ 0\ 0\ 0.0421\ 0\ 0.3316\ 0\ 0.1579\ 0\ 0.2684\ 0\ 0.2895\ 0\ 0.1105\ 0.1789\ 0]^\top$ is optimal (CVX–tested).

7 Conclusions

We have considered the flow control problem for a class of compartmental systems and investigated different saturated control strategies that are decentralized in the sense of networks: to decide its flow, each control agent can rely only on the state variables of the nodes to which it is directly connected. We have provided necessary and sufficient structural conditions for closed–loop asymptotic stability, in terms of connectivity of a network graph. The proposed approach is suitable for generic *nonlinear* and *uncertain* compartmental systems: under suitable assumptions on the network



(a) $u = \mathrm{sat}(-\gamma B^\top x)$ (b) $u = \mathrm{sat}(-\gamma B^\top H x)$ (c) $u = \mathrm{sat}(-\gamma \tilde{B}^\top x)$

Figure 7. Simulations of the network in Fig. 5a. Given $\gamma = 0.01$, $d = [0.6 \;\; 0.2 \;\; 0.7 \;\; 1.8 \;\; 1.2]^\top$ and common random initial conditions, the three considered control strategies are compared: the evolution of u is in the above row, of x in the bottom row. For all the macro–nodes $k \in \mathscr{M} = \{A, B, C, D, E\}$, $\alpha_{kk'} = 0.05$ and $\alpha_{*k} = 0.25$ ($*$ denotes any macro–node in $\mathscr{M}$ suitably connected to k). For macro–nodes A and C, $\alpha_{A*} = \alpha_{C*} = 1$. For macro–nodes B, D and E, $\alpha_{B*} = \alpha_{D*} = \alpha_{E*} = 0.05$; then, at $t = 2000$, $\alpha_{BB} = \alpha_{DE} = \alpha_{EB} = 1$.

topology, stabilization can be achieved robustly (nothing is known about the system functions, apart from smoothness and monotonicity requirements) in a decentralized way.
Moreover, when the overall system is composed of independent compartmental subsystems, we have shown that a particular network–decentralized saturated control strategy, based on the feedback of the total amounts of resource in the subsystems, is asymptotically optimal in terms of the Euclidean norm of the controlled flow.
Among the limits of the approach, we have considered a stabilizability problem and we have not been concerned with set–point regulation, an important issue for compartmental systems. We believe that our techniques can be successfully combined with existing results, such as those by Lee & Ahn (2015); Haddad, Hayakawa, & Bailey (2006). Future research directions can explore the possibility of exactly reaching the desired set–point, at least for some of the variables, by suitably equipping the control arcs with integrators.

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