MIMO Networks with Heterogeneous Uncertainties: Topology-Independent Robust Stability and α-Convergence

Carlos Andres Devia [1] and Giulia Giordano [2]

***Abstract*— We consider network systems where the node dynamics are described by identical MIMO LTI subsystems with transfer-function matrix $F(s)$, while the dynamic interactions associated with the bidirectional arcs are described by identical MIMO LTI subsystems with transfer-function matrix $G(s)$; the dynamics of the individual nodes and arcs are affected by heterogeneous, norm-bounded uncertainties. We provide a topology-independent condition for the robust stability of all possible network systems with a maximum connectivity degree, regardless of their size and interconnection structure. We also give a topology-independent condition that robustly guarantees not only stability, but also α-convergence (i.e. all poles having real part less than a negative $-\alpha$). The proposed frequency-domain conditions are scalable and can be evaluated locally, also for large-scale networks where nodes and arcs can be added or removed in real time. The conditions are applied to assess the robust α-convergence of a suspension bridge system of arbitrary size.**

I. Introduction and Motivation

Network systems, formed by the interconnection of several subsystems, emerge naturally in a wide variety of fields, ranging from smart grids [1] to biological models [2] and groups of people in opinion dynamics [3]. Complex networked system are often analysed by studying the individual subsystem dynamics and the interconnection topology. As with any dynamic system, stability is a fundamental property of interest. Assessing it by exploiting the network structure of the overall system gives rise to the question: *under which conditions does the stability of the individual subsystems guarantee the stability of the whole network?*

Sufficient conditions for the stability of heterogeneous single-input-single-output (SISO) linear time-invariant (LTI) systems are proposed in [4] using the multivariable Nyquist criterion [5]; this type of conditions is further refined in [6], including a partial extension to interconnections of multiple-input-multiple-output (MIMO) systems. The MIMO case is fully addressed in [7], [8] using integral quadratic constraints and in [9] with frequency-domain approaches. Also an approach based on the generalised frequency variable, proposed in [10], was used to obtain necessary and sufficient stability conditions for networks of homogeneous MIMO LTI systems in [11], where the robustness analysis encompasses heterogeneous systems of almost equal agents.

The above approaches require the knowledge of the network topology, which is typically not available for complex and large-scale networks: in such cases, stability conditions that do not depend on the topology would be useful.

Topology-independent sufficient conditions and necessary conditions for robust stability have been provided, based on Nyquist-type arguments, for networks of identical SISO LTI systems with homogeneous [12] and with heterogeneous uncertainties [13]. For interconnections of uncertain MIMO systems, topology-independent robust stability conditions have been recently proposed in [14], which gives a sufficient and a necessary condition for robust stability in the time domain; in the frequency domain, the preliminary results in [15] yield a sufficient robust stability condition for networks of identical MIMO systems with *homogeneous* uncertainties.

Here we consider networks of identical MIMO systems affected by *heterogeneous* uncertainties, in the presence of bidirectional interactions, and we provide frequency-domain sufficient conditions for

- robust stability;
- robust α-*convergence*, i.e. robust stability with largest real part of the poles less than $-\alpha < 0$.

The conditions are topology-independent (they only require the maximum connectivity degree and an upper bound for the Laplacian spectral radius), scalable (they can be evaluated locally) and simple to compute, which makes them useful in large-scale networks and in situations where nodes and arcs can be added or removed from the network in a plug-and-play fashion [16], [17].

After formulating the problem and providing some technical preliminary results in Section II, we present in Section III our main results for topology-independent robust stability, in Theorem 5, and α-convergence, in Theorem 6. Section IV discusses the application example of a suspension bridge system, and exploits Theorem 6 to guarantee robust α-convergence regardless of the system size.

Definitions and Notation. A directed graph (*digraph*) with N nodes and M arcs is a pair $\mathscr{G} = (\mathscr{N}, \mathscr{A})$ where $\mathscr{N} = \{1,\dots,N\}$ denotes the node set and the arc set $\mathscr{A} \subset \mathscr{N} \times \mathscr{N}$ has M elements; $(i,j) \in \mathscr{A}$ if there is an arc from node $i \in \mathscr{N}$ to node $j \in \mathscr{N}$. A digraph is *bidirectional* if $(i,j) \in \mathscr{A}$ implies $(j,i) \in \mathscr{A}$ for all node pairs (i,j). The *degree* $\mathfrak{d}_i$ of node $i \in \mathscr{N}$ is the number of arcs that enter or leave node i. The *maximum connectivity degree* $\mathfrak{D}$ of the graph $\mathscr{G}$ is the maximum degree of its nodes: $\mathfrak{D} = \max_{k \in \mathscr{N}} \mathfrak{d}_k$.

The *incidence matrix* $B \in \{-1,0,1\}^{N \times M}$ is defined as

$$[B]_{ih} = \begin{cases} 1, & \text{if the arc } h=(j,i) \text{ enters node } i, \\ -1, & \text{if the arc } h=(i,j) \text{ leaves node } i, \\ 0, & \text{otherwise.} \end{cases}$$

The work of CAD was supported by the Delft Technology Fellowship grant awarded to GG.

[1]Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands. E-mail: c.a.deviapinzon@tudelft.nl

[2] Department of Industrial Engineering, University of Trento, Trento, Italy. E-mail: giulia.giordano@unitn.it

For bidirectional graphs, the *Laplacian matrix* is $L = BB^\top$.

The space of stable, linear, time invariant and continuous-time transfer functions is denoted by $\mathscr{H}$, while the space of $q \times m$ matrices with entries in $\mathscr{H}$ is $\mathscr{H}^{q\times m}$.

The 2-norm of the real matrix X is denoted as $\|X\| = \sup_{v\neq 0} \|Xv\|/\|v\| = \sqrt{\lambda_{max}(X^\top X)}$, where $\lambda_{max}(S)$ denotes the largest eigenvalue of a symmetric matrix S. Then,

$$\|X \otimes Y\| = \|X\|\|Y\| \quad \text{and} \quad \|XY\| \leq \|X\|\|Y\|, \tag{1}$$

where $\otimes$ is the Kronecker product of matrices [18]. Also, $\|X\| = \|X^\top\|$. Given a square matrix X, its *spectrum* is denoted by $\sigma(X)$ and its *condition number* by $\mathscr{K}(X) \doteq \|X\|\|X^{-1}\|$.

II. Problem Statement and Preliminary Results

Given a network of N nodes and M arcs, represented by the *bidirectional* digraph $\mathscr{G}$, assume that each node (resp. arc) is associated with a stable MIMO linear subsystem represented by the transfer-function matrix $F(s) \in \mathscr{H}^{r\times n}$ (resp. $G(s) \in \mathscr{H}^{n\times r}$), which describes its *nominal* dynamics (n and r are the number of inputs of the nodes and arcs respectively). Furthermore, the generic ith node (resp. hth arc) dynamics is affected by the uncertainty $\Delta_{F_i}(s) \in \mathscr{H}^{r\times n}$, $i \in \{1,\dots,N\}$ (resp. $\Delta_{G_h}(s) \in \mathscr{H}^{n\times r}$, $h \in \{1,\dots,M\}$).

Let vectors $Y_i(s)$ and $U_h(s)$ represent the output of the ith node and hth arc respectively. The dynamics of node i is

$$Y_i(s) = [F(s) + \Delta_{F_i}(s)] \sum_{h=1}^{M} [B]_{ih} U_h(s), \quad i \in \mathscr{N},$$

while the *bidirectional* dynamics of arc $h = (i,j) \in \mathscr{A}$ is

$$U_h(s) = [G(s) + \Delta_{G_h}(s)][Y_i(s) - Y_j(s)], \quad h \in \mathscr{A}.$$

We can stack the output and input vectors as $Y(s) = [Y_1(s)^\top, \dots, Y_N(s)^\top]^\top$ and $U(s) = [U_1(s)^\top, \dots, U_M(s)^\top]^\top$ and write the complete system dynamics as

$$Y(s) = [(I_N \otimes F(s)) + \mathscr{D}_F(s)](B \otimes I_n)U(s), \tag{2}$$
$$U(s) = -[(I_M \otimes G(s)) + \mathscr{D}_G(s)](B^\top \otimes I_r)Y(s), \tag{3}$$

where I_k denotes the identity matrix of size k, while $\mathscr{D}_F = \mathrm{diag}(\Delta_{F_i})_{i=1}^N$ and $\mathscr{D}_G = \mathrm{diag}(\Delta_{G_h})_{h=1}^M$. Hence, the characteristic polynomial of the complete network is

$$\begin{aligned} p(s) &= \det\Big(I_{Nr} + [(I_N \otimes F(s)) + \mathscr{D}_F(s)](B \otimes I_n) \\ &\qquad [(I_M \otimes G(s)) + \mathscr{D}_G(s)](B^\top \otimes I_r)\Big) \\ &= \det\Big(I_{Nr} + L \otimes H(s) + \mathscr{D}(s)\Big), \end{aligned} \tag{4}$$

where $L = BB^\top$, $H(s) = F(s)G(s)$ and $\mathscr{D} = (B \otimes F)\mathscr{D}_G(B^\top \otimes I_r) + \mathscr{D}_F(B \otimes I_n)\mathscr{D}_G(B^\top \otimes I_r) + \mathscr{D}_F(BB^\top \otimes G)$.

We assume stability of the nominal local transfer-function matrices.

Assumption 1: All poles of the transfer-function matrix $H(s) = F(s)G(s)$ have negative real part.

Let $\sigma(H(s)) = \{\lambda_i(s)\}_{i=1}^r$ be the eigenvalues of the transfer-function matrix $H(s)$, which are generic (non-rational in general) complex functions of the variable s: the poles of $\lambda_i(s)$ are not the roots of a polynomial but the set of complex numbers $\tilde{p} \in \mathbb{C}$ such that $\lambda_i^{-1}(\tilde{p}) = 0$. Then, the following result is proven in [15].

Theorem 1: Consider the transfer-function matrix $H(s) \in \mathscr{H}^{r\times r}$ and its eigenvalues $\{\lambda_i(s)\}_{i=1}^r$. Let $\tilde{p} \in \mathbb{C}$ be a pole of the complex function $\lambda_i(s)$, for some $i \in \{1,\dots,r\}$. Then, $\tilde{p}$ is a pole of the transfer-function matrix $H(s)$.

If Assumption 1 is satisfied, then Theorem 1 guarantees that the complex functions $\lambda_i(s)$ are stable (their poles have negative real part). The poles of the transfer-function matrix $H(s)$, i.e. the roots of the denominator polynomial, are much easier to compute than the poles of the generic complex functions $\lambda_i(s)$.

We also assume that $H(s)$ is diagonalisable.

Assumption 2: The transfer-function matrix $H(s)$, with eigenvalues $\sigma(H(s)) = \{\lambda_i(s)\}_{i=1}^r$, can be diagonalised by matrix $V(s)$, so that $V(s)^{-1}H(s)V(s) = \mathrm{diag}(\lambda_i(s))_{i=1}^r$.

Assumption 2 is always satisfied for the important classes of MISO and SIMO systems: if $F(s)$ is a row vector and $G(s)$ is a column vector, then $H(s)$ is a scalar function; if $F(s)$ is a column vector and $G(s)$ is a row vector, then $H(s)$ is a rank-one matrix, hence it is diagonalisable.

Assumption 3: $\|F(j\omega)\| \neq 0$ and $\|G(j\omega)\| \neq 0$ $\forall \omega \in \mathbb{R}^+$.

Under the above assumptions, we wish to assess the robust stability of the class of systems with characteristic polynomial (4), in the presence of *heterogeneous* uncertainties. We look for *topology-independent* conditions, which hold for a whole class of topologies with a given maximum connectivity degree $\mathfrak{D}$ and exclusively depend on local information.

First, some technical preliminary results are needed.

Spectral properties of the Laplacian matrix. For bidirectional digraphs, L is a symmetric matrix that can be diagonalised by unitary matrices W such that $W^{-1}LW = \mathrm{diag}(\gamma_k)_{k=1}^N$, where $\{\gamma_k\}_{k=1}^N$ are the real eigenvalues of L. For unitary matrices, the condition number is one: $\mathscr{K}(W) = 1$. By the Gershgorin circle theorem, the real eigenvalues of L are located inside a circle of radius $\mathfrak{D}$ with centre in $(\mathfrak{D}, 0)$, since $\mathfrak{D}$ is the maximum element along the diagonal of L. Hence,

$$\sigma(L) \subset \{z \in \mathbb{R} : 0 \leq z \leq 2\mathfrak{D}\}. \tag{5}$$

This implies that

$$\|B\|\|B^\top\| \leq 2\mathfrak{D}, \tag{6}$$

since $\|B\| = \|B^\top\| = \sqrt{\lambda_{max}(L)} \leq \sqrt{2\mathfrak{D}}$.

Nominal stability for bidirectional networked systems. A topology-independent condition for the stability of bidirectional network systems without uncertainties is proven in [15].

Theorem 2: Given the networked system (4) with $\mathscr{D} = 0$, under Assumption 1, stability is ensured for *all* network topologies with maximum connectivity degree $\mathfrak{D}$ if, for all $i \in \{1,\dots,r\}$ and $\omega \in \mathbb{R}^+$,

$$\min_{\xi < -(2\mathfrak{D})^{-1}} \{|\lambda_i(j\omega) - \xi|\} > 0, \tag{7}$$

where $\{\lambda_i(j\omega)\}_{i=1}^r = \sigma(H(j\omega))$.

Bauer-Fike Theorem. A well-known result on the spectrum of perturbed matrices is proven in [19].

Theorem 3 (Bauer-Fike Theorem): Let $A, B \in \mathbb{R}^{n\times n}$ with A diagonalisable: $V^{-1}AV = \mathrm{diag}(\lambda_1,...,\lambda_n)$ for some $V \in \mathbb{C}^{n\times n}$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$. For every (complex) eigenvalue β of $A+B$, there exists an index $i \in \{1,\ldots n\}$ such that $|\beta-\lambda_i| \le \mathscr{K}(V)\|B\|$, where $\mathscr{K}(V)$ is the condition number of V.

A Lemma. The following lemma will be used in the proof of Theorem 4.

Lemma 1: Let a, b and u be three complex numbers. If the inequalities $|a-b| \le \varphi$ and $|b-(-u)| > \varphi$ are satisfied for some real $\varphi > 0$, then $a \neq -u$.

III. Stability and α-Convergence

We provide a *topology-independent* condition to assess the robust stability of classes of networked systems associated with *bidirectional* digraphs having maximum connectivity degree $\mathfrak{D}$, where the node and arc dynamics are subject to *heterogeneous* uncertainties. We make the following assumption on the norm bounds for node and arc uncertainties.

Assumption 4: Node and arc uncertainties are bounded as

$$\frac{\|\mathscr{D}_F(j\omega)\|}{\|F(j\omega)\|} \le K_F(j\omega) \quad \text{and} \quad \frac{\|\mathscr{D}_G(j\omega)\|}{\|G(j\omega)\|} \le K_G(j\omega).$$

Equivalently, since $\|\mathrm{diag}(X_i)\| = \max_i \|X_i\|$, we can assume the *local* uncertainty bounds

$$\frac{\|\Delta_{F_i}(j\omega)\|}{\|F(j\omega)\|} \le K_F(j\omega) \quad \forall i \in \{1,\ldots,N\},$$
$$\frac{\|\Delta_{G_h}(j\omega)\|}{\|G(j\omega)\|} \le K_G(j\omega) \quad \forall h \in \{1,\ldots,M\}.$$

We first provide a sufficient condition for robust stability.

Theorem 4: Given the networked system (4) under Assumptions 1, 2 and 3, assume that the nominal system ($\mathscr{D}=0$) is stable for all possible network topologies with maximum connectivity degree $\mathfrak{D}$. Then, stability is robustly guaranteed for the class of uncertain networked systems with maximum connectivity degree $\mathfrak{D}$ that satisfy Assumption 4 if the inequality

$$|\gamma_k\lambda_i(j\omega)+1| > 2\mathfrak{D}\zeta(F,G)K(j\omega) \tag{8}$$

holds for all $k \in \{1,\ldots,N\}$, for all $i \in \{1,\ldots,r\}$, and for all $\omega \in \mathbb{R}^+$, where $\{\lambda_i(s)\}_{i=1}^r$ are the eigenvalues of $H(s)$, while

$$\zeta(F,G) = \mathscr{K}(V(j\omega))\|F(j\omega)\| \ \|G(j\omega)\|, \tag{9}$$
$$K(j\omega) = K_F(j\omega)+K_G(j\omega)+K_F(j\omega)K_G(j\omega). \tag{10}$$

Proof: Denoting by $\beta_q(s)$, $q \in \{1,\ldots,rN\}$, the eigenvalues of the matrix $[(L\otimes H(s))+\mathscr{D}(s)]$, the characteristic polynomial (4) can be rewritten as

$$p(s) = \prod_{q=1}^{rN}[1+\beta_q(s)].$$

By the zero-exclusion theorem [20], since the nominal interconnected system is stable by assumption, robust stability of the uncertain system is guaranteed if, for all possible $\mathscr{D}_F(s)$ and $\mathscr{D}_G(s)$ within the bounds, $p(j\omega)\neq 0$ for all $\omega \in \mathbb{R}^+$, which is equivalent to

$$\beta_q(j\omega) \neq -1, \quad \forall q \in \{1,\ldots,rN\} \quad \text{and} \quad \forall \omega \in \mathbb{R}^+. \tag{11}$$

The eigenvalues of $L\otimes H(s)$ are the products of the eigenvalues of L and of $H(s)$, $\{\gamma_k\lambda_i(s)\}$, and its diagonalisation matrix is $W\otimes V(s)$, where W and $V(s)$ are the diagonalisation matrices for L and $H(s)$ respectively. Hence, in view of the *Bauer-Fike Theorem*, for every $q=1,\ldots,rN$ there is a pair of indices $(k,i) \in \{1,\ldots,N\}\times\{1,\ldots,r\}$ such that

$$|\beta_q(j\omega)-\gamma_k\lambda_i(j\omega)| \le \mathscr{K}(W\otimes V(j\omega))\|\mathscr{D}(j\omega)\|. \tag{12}$$

Using the properties of the 2-norm and of the Kronecker product we have that

$$\begin{aligned}\mathscr{K}(W\otimes V(j\omega)) &= \|(W\otimes V(j\omega))^{-1}\|\|W\otimes V(j\omega)\| \\ &= \|W^{-1}\otimes V^{-1}(j\omega)\|\|W\otimes V(j\omega)\| \\ &= \|W^{-1}\|\|V^{-1}(j\omega)\|\|W\|\|V(j\omega)\| \\ &= \mathscr{K}(W)\mathscr{K}(V(j\omega)) = \mathscr{K}(V(j\omega)).\end{aligned}$$

Also, $\|\mathscr{D}(j\omega)\| \le \|B\|\|B^\top\|\Big(\|F(j\omega)\|\|\mathscr{D}_G(j\omega)\| + \|G(j\omega)\|\|\mathscr{D}_F(j\omega)\| + \|\mathscr{D}_F(j\omega)\|\|\mathscr{D}_G(j\omega)\|\Big) \le \|B\|\|B^\top\|\|F(j\omega)\|\|G(j\omega)\|K(j\omega)$, with $K(j\omega)$ defined in (10). Recall from equation (6) that $\|B\|\|B^\top\| \le 2\mathfrak{D}$ to get

$$\mathscr{K}(W\otimes V(j\omega))\|\mathscr{D}(j\omega)\| \le 2\mathfrak{D}\zeta(F,G)K(j\omega), \tag{13}$$

with $\zeta(F,G)$ defined in (9). Inequalities (12) and (13) yield

$$|\beta_q(j\omega)-\gamma_k\lambda_i(j\omega)| \le 2\mathfrak{D}\zeta(F,G)K(j\omega). \tag{14}$$

Now we can apply Lemma 1 to (14) and (8), setting $\varphi = 2\mathfrak{D}\zeta(F,G)K(j\omega)$, $a=\beta_q(j\omega)$, $b=\gamma_k\lambda_i(j\omega)$ and $u=1$. We get that $\beta_q(j\omega)\neq -1$, hence condition (11) is satisfied and the network system is robustly stable. ■

Remark 1: The stability of the nominal interconnected system, required to apply Theorem 4 (and, as we will see, Theorem 5), can be assessed by checking the *topology-independent* condition in Theorem 2.

The condition (8) in Theorem 4 can be easily checked numerically. However, it is topology-dependent: complete knowledge of the network is required to compute the Laplacian eigenvalues γ_k. The result can be refined, yielding a *topology-independent* condition.

Theorem 5: Given the networked system (4) under Assumptions 1, 2 and 3, assume that the nominal system ($\mathscr{D}=0$) is stable for all possible network topologies with maximum connectivity degree $\mathfrak{D}$. Then, *topology-independent* stability is robustly guaranteed for the class of uncertain networked systems with maximum connectivity degree $\mathfrak{D}$ that satisfy Assumption 4 if the inequality

$$\min_{i\in\{1,\ldots,r\}}\{\phi_i(j\omega)\} > 2\mathfrak{D}\zeta(F,G)K(j\omega) \tag{15}$$

holds for all $\omega \in \mathbb{R}^+$, where

$$\phi_i(j\omega) = \begin{cases} 1 & \text{if } 0 \le \mathrm{Re}(\lambda_i(j\omega)), \\ \frac{|\mathrm{Im}(\lambda_i(j\omega))|}{|\lambda_i(j\omega)|} & \text{if } -\rho|\lambda_i(j\omega)|^2 \le \mathrm{Re}(\lambda_i(j\omega)) < 0, \\ |\rho\lambda_i(j\omega)+1| & \text{if } \mathrm{Re}(\lambda_i(j\omega)) < -\rho|\lambda_i(j\omega)|^2, \end{cases} \tag{16}$$

ρ is an upper bound for the maximum eigenvalue of the Laplacian matrix $L = BB^\top$, $\zeta(F,G)$ is defined as in (9) and $K(j\omega)$ is defined as in (10).

Proof: To show that inequality (15) implies inequality (8) for all $k \in \{1,\dots,N\}$ and $i \in \{1,\dots,r\}$, write the complex number $\lambda_i(j\omega)$ as $\lambda_i = \alpha_i + j\beta_i$, where the argument is omitted for clarity. Now define the convex function

$$D(\gamma_k) = |\gamma_k\lambda_i + 1| = \sqrt{(\gamma_k\alpha_i+1)^2 + \gamma_k^2\beta_i^2}.$$

Taking into account that $\gamma_k \in [0,\rho]$, the minimum of $D(\gamma_k)$ is obtained for $\gamma_k = \gamma_k^*$ given by

$$\gamma_k^* = \begin{cases} 0 & \text{if } 0 \le \alpha_i \\ \frac{-\alpha_i}{|\lambda_i|^2} & \text{if } -\rho|\lambda_i|^2 \le \alpha_i < 0 \\ \rho & \text{if } \alpha_i < -\rho|\lambda_i|^2 \end{cases}$$

In other words, $\gamma_k^* = \arg\min\{D(\gamma_k) \text{ s.t. } \gamma_k \in [0,\rho]\}$. Hence, the minimum value of $D(\gamma_k)$ depends on $\lambda_i(j\omega)$ as follows

1) if $0 \le \alpha_i$, then $D(\gamma_k^*) = 1$,
2) if $-\rho|\lambda_i|^2 \le \alpha_i < 0$, then $D(\gamma_k^*) = \frac{|\mathrm{Im}(\lambda_i(j\omega))|}{|\lambda_i(j\omega)|}$,
3) if $\alpha_i < -\rho|\lambda_i|^2$, then $D(\gamma_k^*) = |\rho\lambda_i(j\omega)+1|$.

Since $|\gamma_k\lambda_i(j\omega)+1| \ge D(\gamma_k^*)$, each case gives a different lower bound for $|\gamma_k\lambda_i(j\omega)+1|$. Thus, by construction, $\phi_i(j\omega)$ satisfies $|\gamma_k\lambda_i(j\omega)+1| \ge \phi_i(j\omega)$. Taking the minimum over all $i \in \{1,\dots,r\}$ makes inequality (15) imply inequality (8) for all $k \in \{1,\dots,N\}$ and $i \in \{1,\dots,r\}$. ■

Theorem 5 avoids the need of computing the eigenvalues of the Laplacian matrix and provides a *topology-independent* condition: all the required information about the topology is the maximum connectivity degree $\mathfrak{D}$ and an upper bound ρ for the maximum Laplacian eigenvalue. The general upper bound provided in (5) is $\rho = 2\mathfrak{D}$, but for many common topologies tighter bounds exist.

A. α-Convergence

So far, the main objective has been to guarantee stability of the overall network. However, in many instances it is desirable to have fast enough convergence, or equivalently an upper bound on the settling time, as per the next definition.

Definition 1: A LTI system with pole set $\mathscr{P}$ is *α-convergent* if $\mathrm{Re}(p) < -\alpha < 0$ for all $p \in \mathscr{P}$.

Based on Theorem 5, we can obtain *topology-independent* sufficient conditions to robustly certify not only stability, but also α-convergence.

Theorem 6: Given the networked system (4) under Assumptions 1, 2 and 3, *topology-independent* α-convergence is robustly guaranteed for the class of uncertain networked systems with maximum connectivity degree $\mathfrak{D}$ that satisfy Assumption 4 if the inequalities

$$\min_{\xi < -(2\mathfrak{D})^{-1}} \{|\lambda_i(j\omega-\alpha) - \xi|\} > 0, \quad i \in \{1,\dots,r\}, \tag{17}$$

$$\min_{i\in\{1,\dots,r\}} \{\phi_i(j\omega-\alpha)\} > 2\mathfrak{D}\widehat{\zeta}(F,G)K(j\omega-\alpha), \tag{18}$$

hold for all $\omega \in \mathbb{R}^+$, with ϕ_i as in (16),

$$\widehat{\zeta}(F,G) = \mathscr{K}(V(j\omega-\alpha))\|F(j\omega-\alpha)\|\,\|G(j\omega-\alpha)\|$$

and K as in (10).

Proof: Take the characteristic polynomial $p(s)$ of the complete network, given in (4), and define $\hat{p}(s)$ as the shifted polynomial $\hat{p}(s) = p(s-\alpha)$. If $\hat{p}(s)$ is stable, then $p(s)$ is α-convergent.

By Theorem 2, inequality (17) guarantees that the nominal networked system associated with $\hat{p}(s)$ is stable, thus Theorem 5 can be applied to check that, if inequality (18) is satisfied, then $\hat{p}(s)$ is stable, hence $p(s)$ is α-convergent. ■

The suitable value of α is problem-dependent and can be selected based on the desired settling time, which can be approximated as $4/\alpha$.

IV. α-Convergence of a Suspension Bridge

For a suspension bridge, it is important not only to ensure stability within suitable uncertainty bounds, but also to guarantee a short enough settling time, so as to prevent long-lasting oscillations.

A suspension bridge can be modelled as a network of interconnected systems (see Fig. 1): the nodes correspond to the cables that hold the bridge road and the arcs correspond to the discretisation of the bridge road that connects the cables. The resulting graph is known as a *ladder graph*, for which the maximum connectivity degree is always $\mathfrak{D} = 3$.

It is worth stressing that the results in this section hold for *any* type of graph with $\mathfrak{D} = 3$, regardless of its topology (information about the topology is not needed).

Fig. 2 shows an example of a ladder graph with 8 nodes and 10 arcs, having incidence matrix

$$B = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{19}$$

Node and arc dynamics. Assume that an affine transformation was used to remove gravitational effects. Then each node can be modelled as a two mass-spring-damper system (see Fig. 3). The state vector is $x = [x_1, x_2, x_3, x_4]^\top$, where x_1 (resp. x_2) corresponds to the displacement of mass m_1 (resp. m_2) from its equilibrium location, while x_3 (resp. x_4) represents the velocity of mass m_1 (resp. m_2). The nodes have a single input u_1, which is a force acting on m_2, and

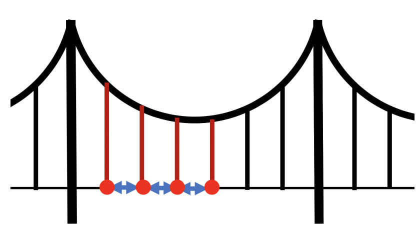

Fig. 1. Suspension bridge (side view). The vertical cables are the nodes (red) and the road discretisation segments are the arcs (blue). The side view shows only one side of the graph. The upper view is similar to the complete graph in Fig. 2.

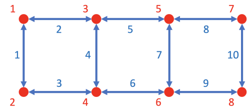

Fig. 2. Example of ladder graph, corresponding to the incidence matrix B in (19). The red dots are nodes and the blue arrows are arcs. The arrows are double headed to denote bidirectional interactions.

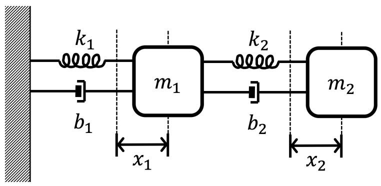

Fig. 3. Simplified dynamic system at the nodes. The cable mass is m_1 and the road mass is m_2; k_i denote spring coefficients and b_i denote damping coefficients. The system is modelled as a double oscillator.

the outputs are the state variables x_2 and x_4. The system matrices are therefore:

$$A_F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{(b_1+b_2)}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{pmatrix} \quad B_F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{pmatrix} \tag{20}$$

$$C_F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad D_F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The arcs can be represented by a mass-spring-damper, with only one mass (M) and two identical dampers (B) and springs (K) at each side. The state variables are the position and velocity of the mass M, while the inputs are the difference between the position and velocity at each side (see Fig. 4). The resulting system matrices are

$$\begin{aligned} A_G &= \begin{pmatrix} 0 & 1 \\ -\frac{2K}{M} & -\frac{2B}{M} \end{pmatrix} & B_G &= \begin{pmatrix} 0 & 0 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix} \\ C_G &= \begin{pmatrix} -2K & -2B \end{pmatrix} & D_G &= \begin{pmatrix} -K & -B \end{pmatrix} \end{aligned} \tag{21}$$

Topology-independent robust α-convergence. We wish to determine the maximum uncertainty magnitude for which the overall bridge system is robustly α-convergent, with $\alpha = 0.4$, regardless of the network size. Using a second order approximation, this means that the settling time is at most $T \approx 4/\alpha = 10$ seconds. To assess topology-independent robust α-convergence, we can apply Theorem 6 with $\rho = 2\mathfrak{D} = 6$.

For the simulation results, we use the parameter values: $k_1 = 200$, $m_1 = 800$, $b_1 = 400$, $k_2 = 200$, $m_2 = 1000$, $b_2 = 800$, $K = 800$, $M = 200$, $B = 800$.

First, we show that condition (17) is satisfied. In this case there are only two eigenvalues λ_i, $i = 1, 2$: λ_1 is zero, because H is a rank-one matrix, thus it satisfies the inequality; Fig. 5 shows the Nyquist plot of λ_2, which is far from the point $-1/(2\mathfrak{D}) = -1/6$. Hence, condition (17) is satisfied.

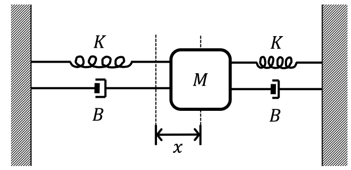

Fig. 4. Simplified dynamic system at the arcs, corresponding to a discretisation of the road. The spring coefficient K and the damping coefficient B account for a rotational spring and damper.

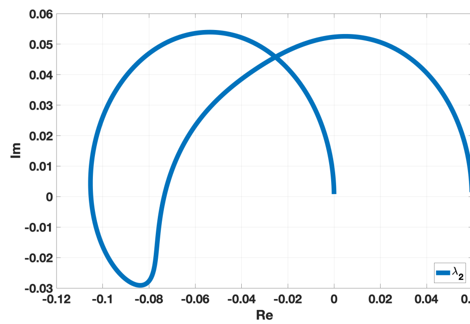

Fig. 5. Nyquist plot of the non-zero eigenvalue $\lambda_2(j\omega - \alpha)$ of $H(j\omega - \alpha)$. Since it is far from the point $-1/(2\mathfrak{D}) = -1/6$, condition (17) is satisfied.

Then, we can determine the upper bound for the uncertainty K that guarantees robust α-convergence for all topologies with $\mathfrak{D} = 3$. Rearranging inequality (18) yields

$$K(j\omega - \alpha) < \theta(j\omega - \alpha) \doteq \frac{\min_{i \in \{1,2\}} \{\phi_i(j\omega - \alpha)\}}{2\mathfrak{D}\widehat{\zeta}(F, G)}. \tag{22}$$

Let us analyse function $\theta(j\omega - \alpha)$ in (22).
On the numerator is $\min\{\phi_1(j\omega - \alpha), \phi_2(j\omega - \alpha)\}$. Since $\lambda_1 = 0$, $\phi_1 = 1$. Regarding λ_2, for different values of ω it satisfies all the cases in (16): Fig. 6 shows the value of $\phi_2(j\omega - \alpha)$, indicating the frequency intervals for each of the cases. From Fig. 6, it can be seen that in this case we have $\min\{\phi_1(j\omega - \alpha), \phi_2(j\omega - \alpha)\} = \phi_2(j\omega - \alpha)$.
The denominator consists of the product $2\mathfrak{D}\mathscr{K}(V(j\omega - \alpha))\|F(j\omega - \alpha)\|\,\|G(j\omega - \alpha)\|$, where $\mathfrak{D} = 3$ for a ladder graph. Fig. 7 shows $\mathscr{K}(V(j\omega - \alpha))$, $\|F(j\omega - \alpha)\|$, $\|G(j\omega - \alpha)\|$, and the product $\widehat{\zeta}(F, G)$ as a function of ω.

As expected, since the input-output matrix $D_G \neq 0$, $\|G\|$ is non-zero at high frequencies. This is reflected also in the frequency response of $\mathscr{K}(V)$. On the other hand, $F(s)$ has a low-pass-filter behaviour, hence $\|F\|$ and $\widehat{\zeta}(F, G)$ tend to zero for high frequencies.

Finally, taking the ratio between ϕ_2 and $2\mathfrak{D}\widehat{\zeta}(F, G)$ gives the upper bound in (22), whose plot is shown in Fig. 8.

According to Fig. 8, the minimum value of the upper bound for $K(j\omega - \alpha)$ is around 0.137, when $\omega = 0.426$ rad/s. This means that, even if the uncertainties were about 6%, i.e. $K_F(j\omega - \alpha) \leq 0.06$ and $K_G(j\omega - \alpha) \leq 0.06$, it would be $K(j\omega - \alpha) \leq 0.1236$; then, $K(j\omega - \alpha)$ would still satisfy the inequality (22) at all frequencies. Hence, the system would be stable and α-convergent with $\alpha = 0.4$. Note that, since $\phi_2 \leq 1$ and $\widehat{\zeta}(F, G)$ has a low-pass-filter response, the uncertainties

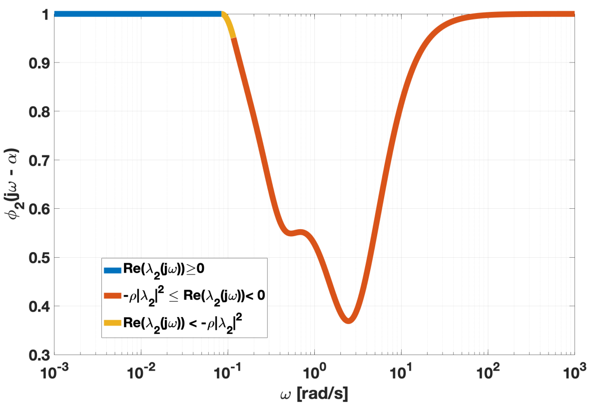

Fig. 6. Plot of $\phi_2(j\omega-\alpha)$; colour denotes the different cases in (16).

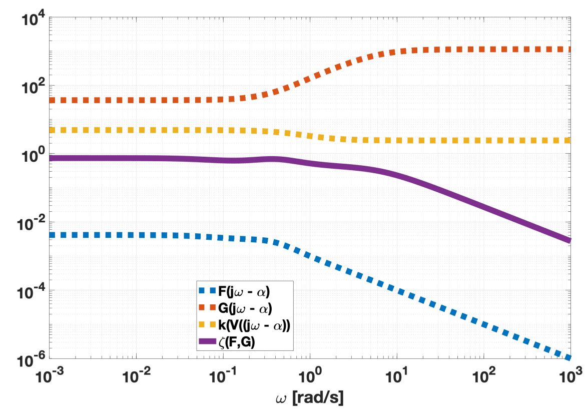

Fig. 7. Frequency response of $\mathscr{K}(V(j\omega-\alpha))$, $\|F(j\omega-\alpha)\|$, $\|G(j\omega-\alpha)\|$, and $\widehat{\zeta}(F,G)$.

can be very large at high frequencies without compromising stability and α-convergence.

Given that the expression of K couples the uncertainties affecting the node transfer function matrix, F, and the arc transfer function matrix, G, it is not possible to immediately deduce separate upper bounds for the individual uncertainties. However, if for instance $K_G \approx 0$, then $K \approx K_F$, which yields an upper bound for the node uncertainties. An analogous result is obtained if $K_F \approx 0$.

V. Conclusions

This paper considered networks of interconnected MIMO systems, where the uncertainties are different for each node and arc. A topology-independent robust stability condition was provided in Theorem 5, and a topology-independent condition to robustly guarantee fast enough convergence was given in Theorem 6. This last result was applied to a suspension bridge system, to determine an upper bound for the uncertainty for which the bridge dynamics exhibit a maximum guaranteed settling time. Future work includes studying how conservative the sufficient conditions are, looking for necessary conditions and extending the results to networks formed by heterogeneous nodes.

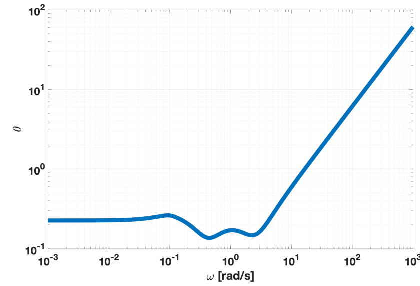

Fig. 8. Largest admissible value for the uncertainty $K(j\omega-\alpha)$ that guarantees robust topology-independent stability, according to equation (22).

References

[1] X. Yu and Y. Xue. Smart grids: A cyber-physical systems perspective. *Proc. IEEE*, 104(5):1058–1070, 2016.

[2] Y. Hori, H. Miyazako, S. Kumagai, and S. Hara. Coordinated spatial pattern formation in biomolecular communication networks. *IEEE Trans. Mol. Biol. Multi-Scale Commun.*, 1(2):111–121, 2015.

[3] B. D. O. Anderson and M. Ye. Recent advances in the modelling and analysis of opinion dynamics on influence networks. *Int. J. Autom. Computing*, 16(2):129–149, 2019.

[4] I. Lestas and G. Vinnicombe. Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems. *IEEE Trans. Autom. Control*, 51(10):1613–1625, 2006.

[5] C. A. Desoer and Y.-T. Wang. On the generalized Nyquist stability criterion. *IEEE Trans. Autom. Control*, 25(2):187–196, 1980.

[6] R. Pates and G. Vinnicombe. Stability certificates for networks of heterogeneous linear systems. In *IEEE Conf. Dec. Control*, pp. 6915–6920, 2012.

[7] S. Z. Khong and A. Rantzer. Scalable stability conditions for heterogeneous networks via integral quadratic constraints. In *Eur. Control Conf.*, pp. 2863–2867, 2014.

[8] M. S. Andersen, S. K. Pakazad, A. Hansson, and A. Rantzer. Robust stability analysis of sparsely interconnected uncertain systems. *IEEE Trans. Autom. Control*, 59(8):2151–2156, 2014.

[9] A. Gattami and R. Murray. A frequency domain condition for stability of interconnected MIMO systems. In *Am. Control Conf.*, vol. 4, pp. 3723–3728, 2004.

[10] S. Hara, T. Hayakawa, and H. Sugata. Stability analysis of linear systems with generalized frequency variables and its applications to formation control. In *IEEE Conf. Dec. Control*, pp. 1459–1466, 2007.

[11] S. Hara, T. Iwasaki, and Y. Hori. Robust stability analysis for lti systems with generalized frequency variables and its application to gene regulatory networks. *Automatica*, 105:96–106, 2019.

[12] F. Blanchini, D. Casagrande, G. Giordano, and U. Viaro. Topology-independent robust stability of homogeneous dynamic networks. *IFAC-PapersOnLine*, 50(1):1736–1741, 2017.

[13] F. Blanchini, D. Casagrande, G. Giordano, and U. Viaro. A bounded complementary sensitivity function ensures topology-independent stability of homogeneous dynamical networks. *IEEE Trans. Autom. Control*, 63(4):1140–1146, 2018.

[14] C. A. Devia and G. Giordano. Topology-independent robust stability conditions for uncertain MIMO networks. *IEEE Control Syst. Lett.*, 5(1):325–330, 2021.

[15] C. A. Devia and G. Giordano. Topology-independent robust stability for networks of homogeneous MIMO systems. *IFAC-PapersOnLine*, 2020.

[16] J. Bendtsen, K. Trangbaek, and J. Stoustrup. Plug-and-play control – Modifying control systems online. *IEEE Trans. Control Syst. Tech.*, 21(1):79-93, 2013.

[17] S. Riverso, M. Farina, and G. Ferrari-Trecate. Plug-and-play decentralized model predictive control for linear systems. *IEEE Trans. Autom. Control*, 58(10):2608-2614, 2013.

[18] P. Lancaster and H. K. Farahat. Norms on direct sums and tensor products. *Mathematics of Computation*, 26(118):401–414, 1972.

[19] F. L. Bauer and C. T. Fike. Norms and exclusion theorems. *Numerische Mathematik*, 2(1):137–141, 1960.

[20] B. Ross Barmish. *New Tools for Robustness of Linear Systems*. Macmilllan, New York, 1994.