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A switched model for mixed cooperative-competitive social dynamics

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***Abstract*— A multi-agent continuous-time nonlinear model of social behaviour allowing for both competition and cooperation is presented and analysed. The state of each agent is represented by its payoff, which the agent aims at maximising. The role of control variables is played by the model parameters, which account for the agents' decisions to either cooperate with or boycott the other agents and can vary in time within assigned intervals. Alliances and enmities can be established at any time, according to either a greedy or a longsighted criterion. The general nonlinear case is first considered. It is proved that, under realistic assumptions, the system evolution is bounded positive (no extinction) and there is a unique globally-stable equilibrium point. As is somehow expected, the optimal decision for all agents corresponds to full cooperation (decision parameters kept at their positive maximum value) in the case of both shortsighted and farsighted criteria. This is not true if some parameters have negative upper bounds (meaning that some agents systematically boycott some others). Then, in the linear case, it is shown that the system is stable for arbitrarily-varying decision parameters, provided that a Metzler matrix associated with full cooperation is Hurwitz. A characterisation of the long-term behaviour of the linear system is also provided. In particular, it is proved that, under stability conditions, a Nash equilibrium exists if a steady strategy is adopted.**

I. INTRODUCTION

A vast literature exists on multi-agent dynamics in social networks, although mostly focused on consensus and cooperation (see, e.g., [16], [24], [25], [26], [27] and bibliographies therein). Yet, in social dynamics, a very important role is played by competition [4], [35]: social agents often have conflicting goals [8], [18], [23], compete for shared resources [7] (a well-studied phenomenon in ecology [3], [32]), or rival for supremacy in social networks [17], [34], races [9], [10], economy [6], [20], [21], [29] and politics [31]. Antagonistic interactions [1], [2], [22] can arise when an agent obstructs or undermines another in the fight for survival or supremacy.

Here, we do not study opinion dynamics, but the dynamics of a group of agents each wishing to maximise its strength by making alliances or undermining rivals. Typically, each agent is allowed to change friends and foes at well-defined time instants based on predefined criteria and on either partial or complete information about the other agents. In most cases the "game rules" remain the same and are of on-off type, that is, alliances and enmities can be fully enabled or disabled. Along the lines of [9], [10], we consider a fairly general kind of interactions that may either continuously change or jump (in particular, switch) at any instant of time. Contrary to [9], [10], no restriction is made on the number of interactions for each agent. Moreover, the individual strategies may be based only on the current state of the network (shortsighted or greedy behaviour) or on its long-term state (farsighted or provident behaviour) strictly related to the existence of stable equilibria. The adopted continuous-time time-variant differential model, whose state variables represent the current agents' strength (payoff), is described in Section II. The decision (or control) variables are embodied by the varying parameters, which belong to assigned intervals.

We first consider the general nonlinear version of the model (Section III), and show that the boundedness and positivity of the solutions can be guaranteed if the nonlinearities are bounded. Proper dominance conditions ensure the existence of stable equilibria. It is proved that, as long as the upper bounds of the decision parameters are positive, full cooperation among agents leads to their maximum payoff, i.e., any other farsighted or shortsighted strategy cannot increase their profit. This property is not true when some decision parameters have a negative upper bound, which means that some agents systematically boycott some others.

We then consider the simple, yet insightful, linear case in which the decision parameters are the coefficients of the state matrix (Section V). It is shown that stability under arbitrary switching is ensured if a Metzler matrix corresponding to the full-cooperation configuration is Hurwitz. Quite remarkably, it is also proved that in the long-run the competition admits at least one Nash equilibrium, where no agent has interest in unilaterally changing its decision if the decisions of the others remain unchanged [23] (Section VI).

Future research directions are suggested in Section VII.

II. MULTI-AGENT MODEL

The state equations adopted to describe the cooperative and obstructive interactions among the interconnected agents have the following general form:

$$\dot{x}_i = -\lambda_i x_i + b_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} \alpha_{ij}\phi(x_i,x_j) - \sum_{\substack{j=1,\\ j\neq i}}^{n} \beta_{ij}\psi(x_i,x_j), \quad (1)$$

for $i = 1,\dots,n$, where:

- the components $x_i(t)$ of the state $x = [x_1, x_2, \dots, x_n]^\top$ represent the current payoff of the i-th agent, whose aim is clearly to increase x_i as much as possible;
- ϕ and ψ are functions (common to all agents) taking values in $\mathbb{R}^+$; $\phi(x_i,x_j)$ and $\psi(x_i,x_j)$ represent, respectively, the cooperative and obstructive action of agent j on agent i;

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- α_{ij} and β_{ij} represent decision parameters whose value depends on agent j;
- λ_i and b_i are positive coefficients characterising the evolution of the i-th agent's state in the absence of interactions.

In all our results, we shall adopt the following **standing assumptions**.

Assumption 1 (Monotonicity): Functions $\phi(x_i, x_j)$ and $\psi(x_i, x_j)$ are increasing with x_j and non-increasing with x_i. $\diamond$

Assumption 2 (Bounds): The decision parameters α_{ij} and β_{ij} in (1), are confined to positive intervals, precisely:

$$0 \leq \alpha_{ij}^- \leq \alpha_{ij} \leq \alpha_{ij}^+ \quad \text{and} \quad 0 \leq \beta_{ij}^- \leq \beta_{ij} \leq \beta_{ij}^+, \tag{2}$$

which are singletons when $\alpha_{ij}^- = \alpha_{ij}^+$ and $\beta_{ij}^- = \beta_{ij}^+$. $\diamond$

Remark 1: When both ϕ and ψ are linear with respect to their arguments, there is no reason to distinguish between ϕ and ψ and it may be assumed, without loss of generality, that $\phi(x_i, x_j) = \psi(x_i, x_j) = x_j$. In this case, the system equations become:

$$\dot{x}_i = -\lambda_i x_i + b_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ij} x_j, \quad i, j = 1, \ldots, n\,, \tag{3}$$

with $a_{ij} = \alpha_{ij} - \beta_{ij}$ and

$$a_{ij}^- \leq a_{ij} \leq a_{ij}^+, \tag{4}$$

where $a_{ij}^- \leq a_{ij}^+$, but a_{ij}^- and a_{ij}^+ have no prescribed sign. $\diamond$

III. Nonlinear Model

This section points out some interesting properties of the nonlinear model (1).

A. Case of bounded nonlinearities

Suppose that functions ϕ and ψ are bounded as

$$0 \leq \phi(x_i, x_j) \leq 1 \quad \text{and} \quad 0 \leq \psi(x_i, x_j) \leq 1\,. \tag{5}$$

Clearly, the upper bound 1 is not restrictive, since a generic upper bound μ could be accommodated by scaling the decision parameters as $\mu\,\alpha_{ij}$, or $\mu\,\beta_{ij}$, and the interaction functions as $\phi(x_i, x_j)/\mu$, or $\psi(x_i, x_j)/\mu$.

Example 1: Condition (5) is satisfied by the functions:

$$\phi(x_i, x_j) = \frac{x_j}{\gamma_\phi x_i + x_j + \delta_\phi}\,, \quad \psi(x_i, x_j) = \frac{x_j}{\gamma_\psi x_i + x_j + \delta_\psi},$$

where γ_ϕ, γ_ψ, δ_ϕ and δ_ψ are strictly positive coefficients. $\diamond$

Proposition 1 (Boundedness): If (5) holds, then the time evolution of system (1) is bounded. $\square$

Proof. Given constraints (2), (5) and λ_i positive, system (1) is the sum of an asymptotically stable linear term $(-\lambda_i x_i)$ and a bounded term; hence the solution is bounded. $\blacksquare$

When the payoff x_i represents the "strength" of agent i, negative values are not meaningful and, therefore, conditions under which the evolution is bounded *and positive (or nonnegative)* must be established. In principle, a trajectory such that $x_i(t_e) = 0$ for some finite t_e (corresponding to *extinction*), could be admissible. However, this scenario would imply that one of the agents vanishes and the system order is reduced of one unit. The next proposition gives a condition under which positivity of the system is guaranteed.

Proposition 2 (Positivity): If

$$b_i \geq \sum_{\substack{j=1,\\ j\neq i}}^{n} \beta_{ij}^+, \tag{6}$$

then the evolution of system (1) is positive for any positive initial condition; hence, no extinction is possible. $\square$

Proof. Let $c_i \doteq b_i - \sum_{j\neq i} \beta_{ij}^+$ for $i = 1, \ldots, n$, and note that $\dot{x}_i \geq -\lambda_i x_i + c_i$. According to the comparison principle, $x_i(t) \geq x_i(0)e^{-\lambda_i t} + c_i(1 - e^{-\lambda_i t})/\lambda$, which is positive if $c_i \geq 0$, namely condition (6) holds, and $x_i(0) > 0$. $\blacksquare$

B. Case of unbounded nonlinearities

If functions ϕ and ψ are unbounded, then the boundedness of the system solutions cannot be guaranteed in general. For example, consider

$$\phi(x_i, x_j) = \psi(x_i, x_j) = \theta(x_i) x_j,$$

with $\theta(x_i)$ non-increasing (quasilinear model) and possibly constant (linear model). In this case, to ensure boundedness, some diagonal dominance conditions are required. To this purpose, assume

$$0 \leq \frac{\partial \phi(x_i, x_j)}{\partial x_j} \leq 1 \quad \text{and} \quad 0 \leq \frac{\partial \psi(x_i, x_j)}{\partial x_j} \leq 1. \tag{7}$$

Again, there is no restriction in assuming the upper bound 1 instead of a generic ν, since it is always possible to normalise the model parameters as $\alpha_{ij}\phi(x_i, x_j) = (\alpha_{ij}\nu)\phi(x_i, x_j)/\nu$. Let us also assume that

$$\phi(x_i, 0) = 0 \quad \text{and} \quad \psi(x_i, 0) = 0,$$

which, together with (7), imply the sector conditions

$$0 \leq \phi(x_i, x_j) x_j \leq x_j^2 \quad \text{and} \quad 0 \leq \psi(x_i, x_j) x_j \leq x_j^2. \tag{8}$$

Theorem 1 (Boundedness and Stability): If, for all i,

$$\lambda_i > \sum_{\substack{j=1,\\ j\neq i}}^{n} \beta_{ij}^+ \quad \text{and} \quad \lambda_i > \sum_{\substack{j=1,\\ j\neq i}}^{n} \alpha_{ij}^+\,, \tag{9}$$

under conditions (7) and (8), the evolution of system (1) is bounded and the systems admits a unique equilibrium point, which is globally stable. $\square$

Proof. The system can be rewritten as

$$\dot{x}_i = -\lambda_i x_i + b_i + \sum_{j\neq i} \underbrace{[\alpha_{ij}\phi(x_i, x_j) - \beta_{ij}\psi(x_i, x_j)]}_{\doteq w_{ij}(x_i, x_j) x_j}$$

where, in view of (7) and (8), $-\beta_{ij}^+ \leq w_{ij}(x_i, x_j) \leq \alpha_{ij}^+$. Then, the system dynamics can be merged into a differential inclusion $\dot{x} = A(w(x))x + b$, with

$$A(w) = \begin{bmatrix} -\lambda_1 & w_{12} & \ldots & w_{1n} \\ w_{21} & -\lambda_2 & \ldots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \ldots & -\lambda_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Since (9) holds and $\lambda_i > 0$, matrix $A(w)$ is strictly row diagonally dominant with negative diagonal entries. Therefore $V(x) = \|x\|_\infty$ is a strong Lyapunov function ensuring the asymptotic stability of the equilibrium $\bar{x} = 0$ if $b = 0$ [15, Proposition 4.57]. Hence, the evolution of the system with the bounded constant term $b \neq 0$ is bounded. Boundedness ensures the existence of an equilibrium point $\bar{x}$ [28], [30].

Consider the system in the shifted variable $z = x - \bar{x}$: $\dot{z}_i = -\lambda_i z_i + \sum \alpha_{ij} \Delta\phi_{ij} - \sum \beta_{ij} \Delta\psi_{ij}$, where $\Delta\phi_{ij} = \phi(z_i + \bar{x}_i, z_j + \bar{x}_j) - \phi(\bar{x}_i, \bar{x}_j)$ and $\Delta\psi_{ij} = \psi(z_i + \bar{x}_i, z_j + \bar{x}_j) - \psi(\bar{x}_i, \bar{x}_j)$. Then, following the approach in [12], [13], [14], we get $\Delta\phi_{ij} = -D_i^{\phi_{ij}} z_i + D_j^{\phi_{ij}} z_j$, where $D_i^{\phi_{ij}} \doteq \int_0^1 \frac{\partial\phi(\sigma z_i + \bar{x}_i, \bar{x}_j)}{\partial x_i} d\sigma > 0$ and $0 \leq D_j^{\phi_{ij}} \doteq \int_0^1 \frac{\partial\phi(\bar{x}_i, \sigma z_j + \bar{x}_j)}{\partial x_j} d\sigma \leq 1$ due to Assumption 1 and condition (7). Similarly, $\Delta\psi_{ij} = -D_i^{\psi_{ij}} z_i + D_j^{\psi_{ij}} z_j$, with $D_i^{\psi_{ij}} > 0$ and $0 \leq D_j^{\psi_{ij}} \leq 1$. The resulting linear differential inclusion is associated with a strongly row diagonally dominant matrix with negative diagonal entries, hence the strong Lyapunov function $V(z) = \|z\|_\infty$ ensures asymptotic stability. Then, uniqueness and global stability of the equilibrium $\bar{x}$ are guaranteed by the results in [12], [13]. ■

Remark 2: When the nonlinearities are unbounded, some variables can become negative due to competition, thus loosing their physical meaning. Positivity can be guaranteed if the model is changed as:

$$\dot{x}_i = \begin{cases} -\lambda_i x_i + b_i + \sum_{j \neq i} \alpha_{ij}\phi - \sum_{j \neq i} \beta_{ij}\psi, & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

IV. Greedy and Farsighted Strategies

To maximise its own payoff, each agent can choose its interactions with the other agents among a set of admissible interactions. Two kinds of decision strategies are considered:

- Instantaneous greedy, or shortsighted, strategy: each agent maximises its own *instantaneous payoff trend*.
- Long-term provident, or farsighted, strategy: each agent maximises its own *long-term payoff*.

In the sequel, each agent is assumed to know the actions of the others as well as the model parameters.

A. Full cooperation

Under conditions that guarantee positivity of the state variables, it can be proved that full cooperation is the most convenient strategy for all agents, as long as it is possible, namely as long as $\beta_{ij}^- = 0$ for all i and all $j \neq i$, which means that all agents can avoid boycotting the others.

Theorem 2 (Full Cooperation): Let system (1) be positive and $\beta_{ij}^- = 0$. For a given positive initial condition $x(0) > 0$, let $x^+(t)$ be the solution corresponding to $\beta_{ij} = \beta_{ij}^- = 0$ and $\alpha_{ij} = \alpha_{ij}^+$. This solution is positive. Moreover, for any other choice of α_{ij} and β_{ij} satisfying (2), the resulting solution $x(t)$ is such that $x_i(t) \leq x_i^+(t)$ for all $i = 1, \ldots, n$ and for all $t \geq 0$. □

Proof. The solution x_i^+ is positive because all terms but $-\lambda_i x_i$ are positive in every equation. Moreover, when $t = 0$,

$$\dot{x}_i - \dot{x}_i^+ = \sum_{\substack{j=1, \\ j \neq i}}^{n} (\alpha_{ij} - \alpha_{ij}^+)\phi(x_i, x_j) - \sum_{\substack{j=1, \\ j \neq i}}^{n} \beta_{ij}\psi(x_i, x_j) \leq 0 \,,$$

hence, in a right neighbourhood of $t = 0$, x_i^+ is larger than, or equal to, x_i. The same argument holds for any time instant $\hat{t} > 0$ such that $x_i(\hat{t}) = x_i^+(\hat{t})$. ■

Remark 3: In the situation considered by Theorem 2, greedy and long-term strategies coincide: as long as the α_{ij}^+ are positive, no agent has the interest of receding from cooperation, because it would immediately see a negative effect. Also, the fully cooperative strategy $\beta_{ij} = \beta_{ij}^- = 0$ and $\alpha_{ij} = \alpha_{ij}^+$ guarantees positivity of the system, so that no extinction can occur. If an agent changes its strategy and boycotts some other agent, then extinctions are possible; yet, no one will benefit from this change. ◇

Since the fully cooperative system is monotone, it enjoys some remarkable properties. Under suitable assumptions, uniqueness and stability of the equilibrium follow from Theorem 1.

Corollary 1 (Boundedness and Stability): Let us consider the fully cooperative system (1), with $\alpha_{ij} = \alpha_{ij}^+$ and $\beta_{ij} = \beta_{ij}^- = 0$. If (7) and (8) hold, and $\lambda_i \geq \sum_{j \neq i} \alpha_{ij}^+$, then the system evolution is bounded and the system admits a unique equilibrium point, which is globally asymptotically stable. □

B. Coalition and competition

Full cooperation, whenever possible, pays off. The best strategy is less obvious when full cooperation is not possible due to fixed boycott actions represented by $\beta_{ij}^- > 0$, which means that agent j always boycotts agent i. In all the scenarios in which some agents collaborate while some others fight, the behaviour of the network of agents can be visualised using a *coalition-competition graph* whose nodes represent the agents and whose arcs represent the interactions, which can be supportive ($\alpha_{ij} > 0$), represented by pointed arrows; obstructive ($\beta_{ij} > 0$), represented by flat-head arrows; or undecided/unknown, represented by dashed lines.

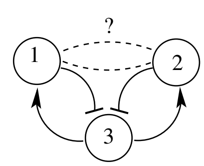

Fig. 1. Coalition-competition graph: agent 3 supports agents 1 and 2, which, however, obstruct agent 3. The interactions between agents 1 and 2 are undecided.

Example 2 (Two agents competing for a resource): Two agents, say 1 and 2, exploit the same natural resource represented by agent 3. Therefore, in Fig. 1, the directed arcs from nodes 1 and 2 to node 3 correspond to obstructions (exploitation), while the arcs from node 3 to nodes 1 and 2 correspond to supportive actions (supply) and are associated with a *fixed* value. As we shall see later, the yet undecided strategies of agents 1 and 2, corresponding to the dashed lines in the aforementioned figure, depend on the bounds and may change in time. Also, for each agent, the short-term strategy may be different from the long-term one. ◇

V. Linear Model

Let the decision parameters a_{ij} in the linear model (3) be bounded as $a_{ij}^- \leq a_{ij} \leq a_{ij}^+$. Negative values correspond to a detrimental effect of j on i, while positive values correspond to a beneficial effect of j on i. Each agent aims at maximising either its own instantaneous payoff trend $\dot{x}_i(t)$ or its asymptotic payoff $x_i(\infty)$ (provided that the system converges). In the first case, the evolution obeys a greedy (shortsighted) criterion, in the second a provident (longsighted) one. The set of all decision parameters a_{ij} satisfying (4) is a $n(n-1)$-dimensional hypercube

$$\mathcal{A} = \prod_{\substack{i,j=1,\dots,n \\ i\neq j}} [a_{ij}^-, a_{ij}^+], \tag{10}$$

where $\prod$ denotes the Cartesian product and $n(n-1)$ is the number of the off-diagonal entries of matrix A. Now, define

$$\bar{a}_{ij} \triangleq \max\{|a_{ij}^-|, |a_{ij}^+|\}, \quad i \neq j, \tag{11}$$

and consider the constant Metzler matrix $\bar{A} \in \mathbb{R}^{n\times n}$ whose entries are

$$\bar{A}_{ij} = \begin{cases} \bar{a}_{ij} & \text{for} \quad i \neq j, \\ -\lambda_i & \text{for} \quad i = j. \end{cases} \tag{12}$$

Note that $\bar{A}$ is not necessarily a compartmental matrix, because diagonal dominance is not enforced.

Assumption 3 (Hurwitz property): Matrix $\bar{A}$ defined in (12) is Hurwitz. ◇

Remark 4: Assumption 3 is motivated by the consideration that, in the symmetric case where $-a_{ij}^- = a_{ij}^+$, $\bar{A}$ is the state matrix corresponding to full cooperation, which is assumed not to lead to divergence. ◇

Theorem 3 (Stability with Arbitrarily Varying Parameters): Under Assumption 3, system (3) with $b_i = 0$, $\forall i$, is stable for arbitrarily varying (possibly switching) $a_{ij}(t)$ that satisfy the bounds (4) at all times t. □

Proof. Since $\bar{A}$ is Hurwitz, there exists a diagonal matrix D such that $\widehat{A} = D^{-1}\bar{A}D$ is strictly row diagonally dominant (see [15, Section 4.5.5]); hence, it admits the infinity norm as a Lyapunov function. By applying the same transformation to the time-varying $A(t)$, $\widetilde{A}(t) = D^{-1}A(t)D$, the diagonal dominance condition remains valid for all t, because $|\widetilde{a}_{ij}(t)| \leq |\widehat{a}_{ij}|$, $\forall t$, $i \neq j$, and the infinity norm is still a Lyapunov function. ■

Therefore, the solutions of system (3) with $b_i \neq 0$ are bounded for arbitrarily varying (possibly switching) $a_{ij}(t)$.

The following result on full cooperation is an immediate consequence of Theorem 2.

Proposition 3 (Full cooperation): If $a_{ij}^+ \geq 0$ for all $i, j \neq i$, the optimal decision for all agents is given by

$$a_{ij}(t) = a_{ij}^+ \tag{13}$$

in the case of both shortsighted and farsighted criteria. □

When $a_{ij}^+ < 0$ for some $i \neq j$, namely some agents systematically boycott other agents, it is in general unclear whether cooperation is more beneficial than obstruction, and the greedy and farsighted strategy of each agent can differ.

Example 3: Consider the network in Example 2. The corresponding state equations can be written as $\dot{x} = Ax + b$, with $x = [x_1, x_2, x_3]^\top$ and

$$A = \begin{bmatrix} -\lambda_1 & a_{12} & r \\ a_{21} & -\lambda_2 & s \\ -p & -q & -\lambda_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The *fixed* negative entries $-p$ and $-q$ represent the exploitation due to agents 1 and 2 of the resources of agent 3 (the environment), while the *fixed* positive entries r and s represent the beneficial effect of agent 3 on 1 and 2. Instead, parameters $a_{12}(t)$ and $a_{21}(t)$ are decision variables that can take positive or negative values, i.e., $a_{ij}^- < 0$ and $a_{ij}^+ > 0$. Two questions arise naturally: Will agents 1 and 2 cooperate? Will the farsighted and shortsighted strategies be different? To determine the best farsighted strategy, consider the steady-state payoff of agent 1 for *constant* values of a_{12} and a_{21} (steady strategy):

$$x_1^\infty = \frac{b_1(\lambda_2\lambda_3+qs) + b_2(\lambda_3 a_{12}-qr) + b_3(a_{12}s+r\lambda_2)}{\lambda_1\lambda_2\lambda_3+pr\lambda_2+qs\lambda_1+a_{12}sp+a_{21}rq-\lambda_3 a_{12}a_{21}}$$

which for p, q, r and s equal to zero becomes

$$x_1^\infty|_{p=q=r=s=0} = \frac{b_1\lambda_2 + b_2 a_{12}}{\lambda_1\lambda_2 - a_{12}a_{21}}.$$

Hence, when p, q, r and s are zero or small, agent 1 benefits from cooperation, and its maximum profit is achieved for $a_{12} = a_{12}^+$ and $a_{21} = a_{21}^+$. The same result holds for agent 2. Assume instead that the four prefixed coefficients are not small. From the previous expression of x_1^∞ it follows that

- if agent 2 fully favours agent 1, i.e., $a_{12} = a_{12}^+$, then agent 1 benefits from cooperating with agent 2 if $\lambda_3 a_{12} > rq$ and hence $a_{21} = a_{21}^+$, otherwise its best strategy will correspond to $a_{21} = a_{21}^-$;
- if $a_{12} < 0$, namely agent 2 exerts an obstructive action on agent 1, then the best choice for agent 1 will be to boycott in turn agent 2.

By taking into account the role of natural resources played by agent 3, we conclude that boycott is the best choice for agent 1 if either it is boycotted by agent 2 or the product qr is high. Clearly, this choice depends also on λ_3: a high value of λ_3 implies a strong feedback action from 3. The above strategy based on the steady state could be different from the best strategy according to a shortsighted criterion. Indeed, for certain values of the current state a *temporary* cooperation between agents 1 and 2 might be preferable, even though in the long run the two agents will end up boycotting each other. Consider, for instance, the case in which $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 0.5$ $r = 2$, $s = 1$, $p = q = 10$, $b = [1\ 1\ 10]^\top$, $a_{12} \in [-1, 1]$ and $a_{21} \in [-1, 1]$ and the initial state is $[1\ 1.2\ 2]^\top$. The blue curves in Fig. 2 represent the evolution of x_1, x_2 and x_3 when x_1 and x_2 cooperate: this policy leads to a rapid exploitation of the environment (bold blue line) which leads also to a temporary extinction of x_3. The red curves represent the state evolution under competition: this policy leads to a milder exploitation of the environment, leading to an extinction whose duration is shorter. The most profitable short-term strategy for both x_1 and x_2 is cooperation (approximately till $t = 0.75$), whereas the most profitable long-term strategy is competition for x_1 (solid lines) and cooperation for x_2 (dashed lines). $\diamond$

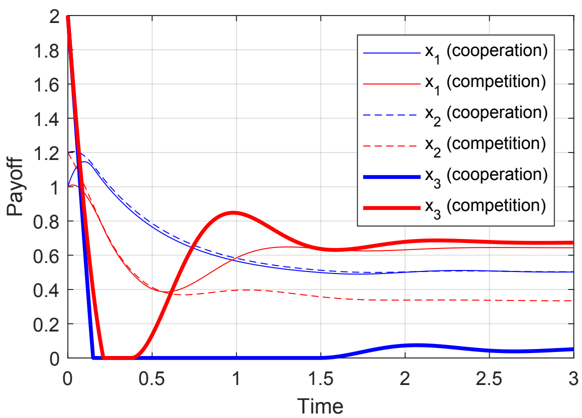

Fig. 2. Time evolution of the system in Example 3 when agents 1 and 2 cooperate (blue curves) and compete (red curves). The initial condition is $x(0) = [1 \ 1.2 \ 2]^\top$ in both cases.

VI. Static Linear Case

This section deals with the long-term steady-state behaviour of the linear system (3) when each agent chooses from the beginning how to permanently interact with the other agents, so that coefficients a_{ij} take on constant values within their admissible closed intervals $[a_{ij}^-, a_{ij}^+]$. Recall that any network configuration, i.e., any set of interactions among the n agents, is represented by a point of the $n(n-1)$-dimensional hypercube $\mathcal{A}$ defined in (10). If A is nonsingular, the steady state of the system is given by

$$\bar{x} = -A^{-1} b. \tag{14}$$

The following proposition provides a necessary and sufficient condition for the steady state (14) to be well defined.

Proposition 4 (Existence of a steady state): A is robustly nonsingular, namely $\det(A) \neq 0$ for all $A \in \mathcal{A}$, if and only if $\det(A)$ has the same sign on all the vertices of $\mathcal{A}$. $\square$

Proof. Since $\det(A)$ is a multi-affine function of the parameters a_{ij}, it reaches its extreme values on the vertices [5]. Hence, if it is positive (respectively negative) at all vertices, it cannot be zero inside the whole hypercube (cf. [11], [12], [19]). On the other hand, suppose that A_1 and A_2 are two vertices of $\mathcal{A}$ such that $\det(A_1) < 0$ and $\det(A_2) > 0$. Then, by continuity of the determinant as a function of the matrix coefficients, there exists a convex linear combination of A_1 and A_2, say A_3, such that $\det(A_3) = 0$. $\blacksquare$

According to Assumption 3, $\bar{A}$ is Hurwitz, which imposes the additional constraint $\operatorname{sgn}(\det(A)) = (-1)^n$ [33].

The following proposition allows us to test whether all the steady-state payoffs are positive by checking only the case in which the decision parameters are all at their extrema.

Proposition 5 (Positivity of the steady state): The steady state $\bar{x}$ defined in (14) is positive for all $A \in \mathcal{A}$ if and only if $-A^{-1}b > 0$ for all the vertices of $\mathcal{A}$. $\square$

Proof. Being $-A^{-1}b$ a multi-affine function of the parameters a_{ij} in the hypercube $\mathcal{A}$, it is positive in the whole hypercube if and only if it is positive at all the vertices (cf. [12], [19]). $\blacksquare$

The next result characterises the individual optimal long-term solution.

Theorem 4 (Extremality): Assume that all agents but one, say agent k, have made their decisions and the only yet undecided parameters are a_{ik}. Then, the maximum steady-state payoff $\bar{x}_k$ of agent k is achieved at an extreme of the range $[a_{ik}^-, a_{ik}^+]$, i.e., by choosing either a_{ik}^- or a_{ik}^+. $\square$

Proof. This proof is given later, after Theorem 5. $\blacksquare$

A. Nash equilibria

An interesting question regarding the linear version of system (1) is whether it admits a long-term Nash equilibrium.

Definition 1: Assume $\det(-A) > 0$ for all possible a_{ij}. A configuration $a_{ij}^* \in [a_{ij}^-, a_{ij}^+]$ for all $i \neq j$ is a *(long-term) Nash equilibrium* if no agent j has interest in unilaterally changing its own strategy, the strategy of the others being fixed: more precisely, for fixed $a_{ik} = a_{ik}^*$, $i, k = 1, \dots, n$, $k \neq i$, agent j does not increase its payoff $\bar{x}_j^\infty$ by choosing $a_{ij} \neq a_{ij}^*$. $\diamond$

Theorem 5 (Nash equilibrium): If $\bar{x} = -A^{-1}b > 0$ for all $A \in \mathcal{A}$, then the system in the static linear case admits a Nash equilibrium. $\square$

Proof. Consider any strategy in $\mathcal{A}$. The long-term steady-state payoff for agent j is, according to Cramer's rule, $\bar{x}_j^\infty = \frac{\nu_j}{\delta}$, with $\delta = \det(-A)$ and $\nu_j = \det(-A_j)$, where matrix $-A_j$ is formed by replacing the jth column of $-A$ by b:

$$\nu_j = \det \begin{bmatrix} \lambda_1 & -a_{12} & \dots & b_1 & \dots & -a_{1n} \\ -a_{21} & \lambda_2 & \dots & b_2 & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & b_n & \dots & \lambda_n \end{bmatrix},$$

$$\delta = \det \begin{bmatrix} \lambda_1 & -a_{12} & \dots & -a_{1j} & \dots & -a_{1n} \\ -a_{21} & \lambda_2 & \dots & -a_{2j} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & -a_{nj} & \dots & \lambda_n \end{bmatrix}.$$

Only the denominator δ of $\bar{x}_j^\infty$ depends on the decision parameters $[a_{1j} \ a_{2j} \ \dots \ a_{nj}]$ pertaining to agent j. By assumption $\delta > 0$, and its value is common to all agents. Let a_{ij}^*, $i, j = 1, \dots, n$, $i \neq j$, be the point where the determinant δ has a minimum. If $a_{ij} = a_{ij}^*$, then no agent has the unilateral interest in changing its strategy (the other decisions being fixed), because this would only decrease its payoff. $\blacksquare$

From the proof of Theorem 5 it follows that agent j, given the choices of all other agents, has to minimise $\delta = \det(-A)$, which is a multi-affine function of the entries a_{ij} of A. Now, a multi-affine function defined on a hypercube reaches its extrema at the vertices, which proves Theorem 4.

The Nash equilibrium is not necessarily unique, as the following example demonstrates.

Example 4 (Multiple Nash Equilibria): For the two-agent system described by the matrices

$$A = \begin{bmatrix} -\lambda_1 & a_{12} \\ a_{21} & -\lambda_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

the long-term payoff of agent 1 is given by

$$x_1^\infty = \frac{b_1\lambda_2 + b_2 a_{12}}{\lambda_1\lambda_2 - a_{12}a_{21}}.$$

If agent 2 boycotts agent 1, namely, $a_{12} = a_{12}^- < 0$, then the most profitable choice for 1 is to retaliate, i.e., to choose $a_{21} = a_{21}^- < 0$. Conversely, if $a_{12} = a_{12}^+ > 0$, then the most profitable choice for 1 is to cooperate: $a_{21} = a_{21}^+ > 0$. Both of these configurations are Nash equilibria. ◇

VII. Conclusions

A continuous-time nonlinear model of social behaviour that considers both cooperative and competitive interactions has been presented and analysed. Time-varying, possibly switching, model parameters represent the agents' decisions. Under realistic assumptions on the interaction relations, initial conditions and exogenous inputs, the network evolution is bounded and positive, and the system admits a unique globally stable equilibrium point (Theorem 1). Independently of the adopted shortsighted or farsighted interaction criterion, full cooperation pays off whenever it is possible (Theorem 2). In the case of *linear interactions*, the system is stable in the absence of exogenous inputs for arbitrarily varying, possibly switching, model parameters (Theorem 3). The existence of a positive steady state of the linear version can be easily checked via a vertex test and the individual optimal strategy is at the extrema of the allowed range (Theorem 4). Finally, at least one Nash equilibrium point exists in the static case when the decision parameters in the linear model are kept constant (Theorem 5).

Possible directions of future research include: (i) the evaluation of the effects of obstructions in the general nonlinear model, (ii) the comparison of different (shortsighted or farsighted) strategies, and (iii) the existence of Nash equilibria in nonlinear models.

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