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# Aggregates of Positive Impulse Response systems: a decomposition approach for complex networks

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*Abstract*— **To simplify the analysis of complex dynamical networks, we have recently proposed an approach that decomposes the overall system into the sign-definite interconnection of subsystems with a Positive Impulse Response (PIR). PIR systems include and significantly generalise input-output monotone systems, and the PIR property (or equivalently, for linear systems, the Monotonic Step Response property) can be evinced from experimental data, without an explicit model of the system. An aggregate of PIR subsystems can be associated with a signed matrix of interaction weights, hence with a signed graph where the nodes represent the subsystems and the arcs represent the interactions among them. In this paper, we prove that stability is *structurally* ensured (for any choice of the PIR subsystems) if a Metzler matrix depending on the interaction weights is Hurwitz; this condition is non-conservative. We also show how to compute an influence matrix that represents the steady-state effects of the interactions among PIR subsystems.**

## I. Introduction

A complex dynamical network can be conveniently studied as the signed interconnection of subsystems with a *Positive Impulse Response* (PIR) and represented by an *aggregate graph*, whose nodes are associated with the subsystems and whose arcs are associated with the signed interactions among them. Based on this observation, in [9] we have adapted the structural classification of oscillatory and multistationary behaviours in biological systems, based on the exclusive or concurrent presence of positive and negative cycles in the aggregate graph, previously proposed in [7] for systems with a sign-definite Jacobian and in [8] for signed interconnections of input-output monotone subsystems. Aggregates of PIR subsystems include and generalise aggregates of input-output monotone subsystems. Positivity of the impulse response can be easily assessed from experimental data, while establishing monotonicity of a system requires its state space description and this can hinder the application of tools from monotone systems theory [3], [4], [23], [24] to realistic biological networks, whose state model is often too complex and plagued by uncertainty [1], [15]. Therefore, we suggest an alternative approach based on positivity of the system impulse response.

In this paper, we further investigate this type of decomposition, briefly presented in Section II, and provide new *structural* (or qualitative [19]) results. We adopt a parameter-free approach [5], [6], [10], [12] that is particularly useful when studying biological systems, due to their intrinsic uncertainty and variability and, at the same time, their extraordinary robustness [1], [16], [17]. Given a linearised model, we partition it into subsystems having a Positive Impulse Response (PIR). A summary of available criteria to establish whether a system has a PIR, which in the linear case is equivalent to having a Monotonic Step Response, is recalled in Section III, along with new observations on how to decompose a system into an aggregate of PIR subsystems. Then, a signed interconnection among PIR subsystems is described by the corresponding aggregate graph, a directed graph whose arcs have weights representing the interaction strengths. The topology of this signed graph represents the system *structure*. We show that interesting structural information can be inferred from this signed graph, without any knowledge about the actual value of the parameters and even about the specific transfer functions.

In Section IV we show that the overall system is *structurally* Hurwitz stable (for any choice of the PIR subsystems) if a certain Metzler matrix, including the maximum interaction weights, is Hurwitz. This condition is not conservative: if this matrix has a positive Frobenius eigenvalue, then the system is unstable for suitable choices of the transfer functions. Then, extending results that we have recently proposed [12] for systems admitting the so-called BDC-decomposition [6], [10], [12], in Section V we analyse the steady-state behaviour of the aggregate system, represented by the *influence matrix* $M$. This signed matrix has entries $M_{ij} \in \{+,-,0,?\}$, depending on the sign of the steady-state variation of the output of the $i$th subsystem due to a persistent input affecting the $j$th subsystem. In particular, an entry is '$+$' if the variation is *structurally* positive (for all possible choices of the PIR subsystems), '$-$' if it is structurally negative, '0' in the case of perfect adaptation and undetermined '?' if the variation depends on the specific parameters and PIR subsystems.

We illustrate the results on examples in Section VI, while in Section VII we draw conclusions.

## II. Aggregates of PIR Subsystems

Given a complex dynamical network, our analysis relies on the linearisation of the model, which is then partitioned into Positive Impulse Response (PIR) subsystems to obtain a simpler aggregate graph of the system.

*Definition 1:* The linear SISO system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$

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is a PIR system if its impulse response is positive,

$$Ce^{At}B \geq 0,$$

for all $t \geq 0$, or equivalently if its step response

$$\int_0^t Ce^{A(t-\tau)}B \, d\tau,$$

is monotonically increasing. ◇

PIR systems are a much wider family than input-output monotone systems: (1)–(2) is *input-output monotone* if

(i) $A$ is Metzler: $A_{ij} \geq 0$ for $i \neq j$;

(ii) $B$ and $C$ are nonnegative.

Any input-output monotone linear system is a PIR system. The opposite is not true: having a linearisation for which (i) and (ii) hold is a stronger requirement.

*Example 1:* [9] Consider the chemical reaction network

$$X_1 + X_2 \xrightarrow{g_{12}} X_3, \; X_3 \xrightarrow{g_3} X_1, \; X_2 \xrightarrow{g_2} \emptyset, \; X_3 \xrightarrow{\tilde{g}_3} \emptyset,$$
$$\emptyset \xrightarrow{x_{1,0}} X_1, \; \emptyset \xrightarrow{x_{2,0}} X_2.$$

The concentrations evolve according to the equations

$$\begin{aligned} \dot{x}_1 &= -g_{12}(x_1, x_2) + g_3(x_3) + x_{1,0} + u \\ \dot{x}_2 &= -g_{12}(x_1, x_2) - g_2(x_2) + x_{2,0} \\ \dot{x}_3 &= +g_{12}(x_1, x_2) - g_3(x_3) - \tilde{g}_3(x_3) \\ y &= x_3, \end{aligned}$$

where the $g$'s and $\tilde{g}_3$ are increasing functions. The linearised system for $x = [x_1 \quad x_2 \quad x_3]^\top$ is

$$\dot{x} = \begin{bmatrix} -\alpha & -\beta & \gamma \\ -\alpha & -(\beta+\delta) & 0 \\ \alpha & \beta & -(\gamma+\epsilon) \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad (3)$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x, \quad (4)$$

where the positive parameters $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ correspond to the partial derivatives. The state matrix of the linearised system is not Metzler. Yet, (3)–(4) is a PIR system [9]. ◇

We consider PIR subsystems whose transfer functions are *admissible*, according to the following definition.

*Definition 2:* Given the impulse response $f(t)$ of a linear SISO system, the Laplace transform

$$F(s) = \mathcal{L}[f(t)] \doteq \int_0^\infty f(t) e^{-st} dt$$

is its *transfer function*. The transfer function

$$F(s) = e^{-s\tau} G(s)$$

is *admissible* if $G$ is rational, strictly proper (hence, $\lim_{s\to\infty} F(s) = 0$) and stable (namely, its poles have negative real part), and the delay is positive, $\tau > 0$. ◇

We have considered a delay term because a (possibly very small) delay is always present in practice. We call *PIR transfer function* a transfer function $F(s)$ corresponding to a positive impulse response $f(t)$.

### A. Interconnections of PIR subsystems

Let $y(s)$ be an $N$-dimensional vector including the Laplace-transformed outputs of the $N$ PIR linearised subsystems that compose the overall system. Our model can be written as

$$y(s) = \Phi(\Sigma, s) y(s), \quad (5)$$

where matrix $\Phi(\Sigma, s)$ has entries of the form

$$\Phi_{ij}(\sigma_{ij}, s) = \sigma_{ij} F_{ij}(s), \quad (6)$$

with $F_{ij}(s)$ admissible PIR transfer functions, and $\Sigma$ is the interaction matrix, whose entries are the coefficients $\sigma_{ij}$, which account for the interactions among subsystems.

Matrix $\Sigma$ is the weighted adjacency matrix of the directed *aggregate graph*, where the nodes represent the (linearised) PIR subsystems. In the graph, there exists an arc from node $j$ to node $i$ if and only if $y_j$ affects $y_i$, namely $\sigma_{ij} \neq 0$. The arc from node $j$ to node $i$ can be either positive or negative, depending on the sign of its weight $\sigma_{ij}$, which encodes the signed interaction between subsystem $j$ and subsystem $i$.

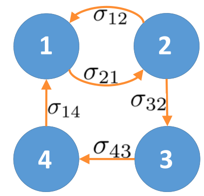

Fig. 1: Aggregate graph of the system with the interaction matrix in (7).

*Definition 3:* Given an aggregate of interconnected subsystems, matrix $S = \text{sign}[\Sigma]$ is the system *structure*, while matrix $\Sigma$ is a *realisation* of structure $S$. ◇

*Example 2:* Consider a system of the form

$$\begin{aligned} &\dot{x}_1 = f_1(x_1, \sigma_{12} y_2, \sigma_{14} y_4), && \dot{x}_2 = f_2(x_2, \sigma_{21} y_1), \\ &\dot{x}_3 = f_3(x_3, \sigma_{32} y_2), && \dot{x}_4 = f_4(x_4, \sigma_{43} y_3), \\ &y_i = g_i(x_i), \quad i = 1,2,3,4, \end{aligned}$$

and its linearisation

$$\begin{aligned} &\dot{x}_1 = A_1 x_1 + B_{12}\sigma_{12} y_2 + B_{14}\sigma_{14} y_4, \\ &\dot{x}_2 = A_2 x_2 + B_{21}\sigma_{21} y_1, \quad \dot{x}_3 = A_3 x_3 + B_{32}\sigma_{32} y_2, \\ &\dot{x}_4 = A_4 x_4 + B_{43}\sigma_{43} y_3, \quad y_i = C_i x_i, \quad i = 1,2,3,4, \end{aligned}$$

where each subsystem $(A_i, B_{ij}, C_i)$ is a PIR system with input $\sigma_{ij} y_j$ and $\sigma_{ij}$ is the weight of the interaction between subsystems $i$ and $j$. The system *structure* is given by the sign pattern of the interaction matrix

$$\Sigma \doteq \begin{bmatrix} 0 & \sigma_{12} & 0 & \sigma_{14} \\ \sigma_{21} & 0 & 0 & 0 \\ 0 & \sigma_{32} & 0 & 0 \\ 0 & 0 & \sigma_{43} & 0 \end{bmatrix}, \quad (7)$$

associated with the graph in Fig. 1. Then, in the Laplace domain, the linearised system corresponds to

$$\begin{aligned} y_1(s) &= F_{12}(s)\sigma_{12} y_2(s) + F_{14}(s)\sigma_{14} y_4(s), \\ y_2(s) &= F_{21}(s)\sigma_{21} y_1(s), \\ y_3(s) &= F_{32}(s)\sigma_{32} y_2(s), \\ y_4(s) &= F_{43}(s)\sigma_{43} y_3(s), \end{aligned}$$

where $f_{ij}(t) = \mathcal{L}^{-1}[F_{ij}(s)]$ are PIR. ◇

Given a signed interconnection of PIR subsystems, we consider the following class of problems: *if $F_{ij}$ are generic PIR transfer functions and the sign pattern $S = sign[\Sigma]$ is known, what can we infer about the overall system?*

## III. Properties of PIR Transfer Functions and Decomposition

Determining if a transfer function corresponds to a positive impulse response is a well-studied problem and partial results are available [14], [18]. Also the link between PIR systems and input-output monotone systems has been investigated. Any input-output monotone linear system has a PIR transfer function. Under proper assumptions, any PIR transfer function admits a positive realisation [11], which is non-minimal: to find a state space representation that is input-output monotone, the state needs to be artificially augmented. This augmentation can be avoided, under some assumptions, by considering eventually positive minimal realisations [2].

*Remark 1: Any* linear system can be seen as an aggregate of PIR subsystems of the form (5)–(6). A trivial PIR decomposition is that into first-order PIR subsystems. Indeed, the equations in the time domain and in the Laplace domain are

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j \ \xrightarrow{\mathcal{L}} \ x_i(s) = \sum_{j \neq i} \frac{a_{ij}}{s - a_{ii}} x_j(s).$$

Let $y(s) \doteq \left[x_1(s) \dots x_n(s)\right]^\top$. Then, we can write the overall system as in (5)–(6), with $\Phi_{ij}(\sigma_{ij}, s) \doteq a_{ij}\frac{1}{s-a_{ii}}$, hence $\sigma_{ij} \doteq a_{ij}$ and $F_{ij} \doteq \frac{1}{s-a_{ii}}$. Here, $\Sigma \in \mathbb{R}^{n\times n}$. $\diamond$

A significant decomposition is achieved if we reduce the size of $\Sigma$ and aggregate more variables into a single PIR subsystem. To find a non-trivial PIR decomposition, we need *a priori* information on the system, in particular concerning its zeros and poles. We can rely on a list of properties [9] that ensure a PIR behaviour in terms of transfer functions.

$F(s)$ is a PIR transfer function

(a) only if it has no complex dominant poles;
(b) if it is the positive feedback of a PIR system;
(c) if it is the cascade of PIR systems;
(d) if it has $n$ real poles and no zeros;
(e) if it has $n$ real poles and $m < n$ real zeros with the ordering property $-p_1 > -z_1$, $-p_2 > -z_2$, $\dots -p_m > -z_m$, while the other real poles are arbitrary [14], [18].

*Example 3:* Consider the matrix

$$A = \left[\begin{array}{ccc|cc|cc|c} - & - & 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{14} \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ 0 & + & - & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \sigma_{21} & - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & - & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sigma_{32} & - & + & 0 \\ 0 & 0 & 0 & 0 & 0 & - & - & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{43} & - \end{array}\right],$$

where we assume that all square diagonal blocks have real negative eigenvalues. Then, the overall system is an aggregate of four stable PIR systems, with state dimensions $n_1 = 3$, $n_2 = 2$, $n_3 = 2$ and $n_4 = 1$, regardless of the choice of the signed nonzero entries. In fact, the subsystem associated with the first block, represented by the matrices

$$A_1 = \begin{bmatrix} -\alpha & -\beta & 0 \\ \gamma & -\delta & -\epsilon \\ 0 & \phi & -\mu \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

is a PIR system in view of condition (d), since the corresponding transfer function has no zeros (and has real negative poles by assumption). The second subsystem is monotone, hence it is a PIR system. The third block is associated with a transfer function with no zeros (and real negative poles), hence it is a PIR system based on condition (d). The fourth block is a first-order system, hence it is a PIR system. The corresponding matrix $\Sigma$ has the same form as in (7). $\diamond$

To write the system in the form (5)–(6), the most convenient experimental-based approach is to consider a macroscopic model where the interactions among the PIR transfer functions are qualitatively expressed.

## IV. Structural Stability of PIR Aggregates

We provide a *structural* stability analysis of PIR aggregates, exclusively based on the following qualitative information (which can be reasonably deduced by experiments):

- all the transfer functions $F_{ij}(s)$ are Hurwitz stable and PIR (and can include arbitrary delays);
- the sign and the maximum value $w_{ij}$ of all the interaction weights are known:

$$|\sigma_{ij}| \leq w_{ij}; \tag{8}$$

- the static gain of each transfer function is bounded as

$$F_{ij}(0) \leq \phi_{ij}.$$

The next well-known result is very useful (see, for instance, [22]) and shows an important property: the zero frequency amplitude is a bound for all frequency amplitudes.

*Proposition 1:* If $F(s)$ is a PIR transfer function, then

$$|F(\jmath\omega)| \leq F(0), \quad \text{for all } \ \omega \geq 0.$$

$\square$

*Proof:*

$$\begin{aligned} |F(\jmath\omega)| &= \left|\int_0^\infty f(t)e^{\jmath\omega t}\, dt\right| \leq \int_0^\infty \left|f(t)e^{\jmath\omega t}\right|\, dt \\ &= \int_0^\infty |f(t)|\, dt = \int_0^\infty f(t)\, dt = F(0). \end{aligned}$$

$\blacksquare$

Without restriction, since $F_{ij}(0) \neq 0$ for a PIR system in view of Proposition 1, we can scale the transfer functions as $F_{ij}(s) := F_{ij}(s)/F_{ij}(0)$ and include the static gains in the interaction coefficients $\sigma_{ij} := \sigma_{ij}F_{ij}(0)$. Hence, without loss of generality, we have the following standing assumption.

*Assumption 1:* For all $i$ and $j$,

$$F_{ij}(0) = 1. \tag{9}$$

$\diamond$

By assumption, all transfer functions are stable, therefore instability can be caused by the interconnections only. Our stability analysis must then take into account the system interconnection topology, given by matrix $\Sigma$.

We consider the following three matrices:

- the first is
$$[-I+\Sigma] = [-I + \Phi(\Sigma, 0)];$$
- the second is
$$[-I+Z], \quad Z_{ij} \doteq z_{ij} \in \mathbb{C}, \quad |z_{ij}| \leq w_{ij},$$
where matrix $Z$ has the same pattern as $\Sigma$, but the entries $\sigma_{ij}$ are replaced by complex numbers $z_{ij}$ such that $|z_{ij}| \leq w_{ij}$;
- the third is
$$[-I+\Omega], \quad \Omega_{ij} = w_{ij},$$
where the Metzler matrix $\Omega$ has the same pattern as $\Sigma$, but the entries $\sigma_{ij}$ are replaced by their bounds $w_{ij}$.

*Example 4:* For Example 2, whose matrix $\Sigma$ is in (7),

$$[-I+Z] = \begin{bmatrix} -1 & z_{12} & 0 & z_{14} \\ z_{21} & -1 & 0 & 0 \\ 0 & z_{32} & -1 & 0 \\ 0 & 0 & z_{43} & -1 \end{bmatrix} \in \mathbb{C}^{4\times 4} \tag{10}$$

and

$$[-I+\Omega] = \begin{bmatrix} -1 & w_{12} & 0 & w_{14} \\ w_{21} & -1 & 0 & 0 \\ 0 & w_{32} & -1 & 0 \\ 0 & 0 & w_{43} & -1 \end{bmatrix} \in \mathbb{R}^{4\times 4}. \tag{11}$$

$\diamond$

Since $[-I+\Omega]$ is Metzler, it has a dominant real eigenvalue (Frobenius eigenvalue) that we denote as $\lambda_M(\Omega)$,

$$\lambda_M(\Omega) = \max_{\lambda \in \sigma[-I+\Omega]} \mathrm{Re}(\lambda), \quad \lambda_M(\Omega) \in \sigma[-I+\Omega],$$

where $\sigma[-I+\Omega]$ denotes the matrix spectrum.

*Theorem 1:* If $\lambda_M(\Omega) < 0$, then the overall system described by (5)–(6) is Hurwitz stable, for all admissible PIR transfer functions with $F_{ij}(0) = 1$ (and arbitrary delay) and for all $\sigma_{ij}$ that satisfy (8). $\square$

Since we take into account all possible choices of admissible transfer functions, the condition of the theorem is not conservative, as shown next.

*Theorem 2:* If $\lambda_M(\Omega) > 0$, then the overall system described by (5)–(6) is not Hurwitz stable for some choice of admissible PIR transfer functions with $F_{ij}(0) = 1$ (and arbitrary delay) and of $\sigma_{ij}$ that satisfy (8). $\square$

### A. Proof of the theorems

The key idea of the proof is that, if $[-I+\Omega]$ is Hurwitz, then $[-I+Z]$ is Hurwitz, hence nonsingular; this, in turn, implies that the overall system is stable. The result relies on the following lemmas.

*Lemma 1:* The overall system (5)–(6) is Hurwitz stable for all admissible PIR transfer functions with $F_{ij}(0) = 1$ (and arbitrary delay) and all $\sigma_{ij}$ satisfying (8) if

$$\det[-I+Z] \neq 0$$

for all *complex* $z_{ij}$ with $|z_{ij}| \leq w_{ij}$. $\square$

*Proof:* Structural Hurwitz stability is equivalent to the fact that, for all $\sigma_{ij}$ with $|\sigma_{ij}| \leq w_{ij}$, the polynomial

$$p(s) = \det[I - \Phi(\Sigma, s)]$$

has no roots with nonnegative real part. By contradiction, assume that $p(\hat{s}) = 0$, with $\mathrm{Re}(\hat{s}) \geq 0$. Then, we can scale $\Sigma$ as $\alpha\Sigma$, for $0 \leq \alpha \leq 1$, so that the unstable root $\hat{s}$ is on the imaginary axis. This is always possible: indeed, when $\alpha$ is taken small enough, the system becomes Hurwitz stable and the roots of $p(s)$ are continuous functions of $\alpha\Sigma$. Then, $\det[I - \Phi(\alpha\Sigma, \jmath\omega)] = 0$ can be rewritten as

$$\det[-I+Z] = 0,$$

where $z_{ij} = \alpha\sigma_{ij}F_{ij}(\jmath\omega)$ satisfy $|z_{ij}| \leq w_{ij}$, because $|z_{ij}| = |\alpha\sigma_{ij}F_{ij}(\jmath\omega)| \leq |\alpha\sigma_{ij}F_{ij}(0)| = |\alpha\sigma_{ij}| \leq w_{ij}$ in view of Proposition 1. We have thus reached a contradiction. $\blacksquare$

*Lemma 2:* Let $A = -I + Z$ be a complex matrix where $Z$ has zero diagonal entries. If $(-I + |Z|)$ is Hurwitz, where $|Z|$ is the matrix of the moduli, then $A$ is Hurwitz. $\square$

*Proof:* It is known that $A = -I + Z$ is Hurwitz if and only if there exists a small enough $\tau > 0$ such that $I + \tau A$ is Schur. Consider $0 < \tau < 1$. Then, the matrix $I + \tau A = I + \tau(-I + Z) = (1-\tau)I + \tau Z$ is Schur (hence, the proof is concluded) if the matrix of the moduli

$$|(1-\tau)I + \tau Z| = I + \tau(-I + |Z|)$$

is Schur (cf. [13], pag. 404, Exercise 6.2.P4). Since $-I + |Z|$ is Hurwitz, there exists $\tau$ such that this condition holds. $\blacksquare$

*Lemma 3:* The real Metzler matrix $(-I + |Z|)$, where $|z_{ij}| \leq w_{ij}$, is Hurwitz if matrix $[-I+\Omega]$, with $\Omega_{ij} = w_{ij}$, is Hurwitz. $\square$

*Proof:* A Metzler matrix $A$ is stable if and only if, for some positive vector $z$, $z^\top A < 0$ componentwise. If $[-I+\Omega]$ is Hurwitz, then $z^\top[-I + |Z|] < z^\top(-I+\Omega) < 0$ for some positive $z$, since $[-I+\Omega]$ is Metzler as well. Hence, $(-I + |Z|)$ is Hurwitz. $\blacksquare$

We can now prove the main results of the section.

**Proof of Theorem 1.** If $[-I+\Omega]$ is Hurwitz, then $(-I + |Z|)$ is Hurwitz (Lemma 3), therefore the complex matrix $-I + Z$ is Hurwitz (Lemma 2), hence nonsingular. This implies Hurwitz stability of the overall system (5)–(6) (Lemma 1). $\blacksquare$

**Proof of Theorem 2.** If $\lambda_M(\Omega) > 0$, then by continuity there exist complex numbers $z_{ij}$, with $|z_{ij}| \leq w_{ij}$, such that matrix $[-I+Z]$ is singular. For instance, $z_{ij}$ can be of the form $\alpha w_{ij}$, with scaling factor $0 \leq \alpha \leq 1$, so that $Z = \alpha\Omega$. The proof is concluded if we show that there is a choice of admissible PIR transfer functions (according to Definition 2) and of $\sigma_{ij}$ within the bound (8) such that, for the given $z_{ij}$,

$$\sigma_{ij}F_{ij}(\jmath\omega) = \sigma_{ij}G_{ij}(\jmath\omega)e^{-\tau_{ij}\jmath\omega} = z_{ij}.$$

Such functions do exist. Indeed, we can arbitrarily fix $\omega > 0$ and take

$$\sigma_{ij}F_{ij}(\jmath\omega) = \sigma_{ij}\frac{1}{1+\theta_{ij}\jmath\omega}e^{-\tau_{ij}\jmath\omega}.$$

Then, we can always choose $\theta_{ij}$ to adjust the modulus,

$$|\sigma_{ij} F_{ij}(\jmath\omega)| = \sigma_{ij} \frac{1}{\sqrt{1+\theta_{ij}^2\omega^2}} = |z_{ij}|,$$

and $\tau_{ij} > 0$ to adjust the phase, and we have the equality. ■

*Remark 2:* The proposed conditions have a *strength*: they are practically not conservative, because $\lambda_M(\Omega) < 0$ implies structural stability and $\lambda_M(\Omega) > 0$ implies that there is no structural stability. Only the critical case $\lambda_M(\Omega) = 0$ is undetermined. On the other hand, the conditions are delay-independent: their *weakness* is that they are "almost" necessary, since stability is required for any choice of the delay values, which is a strong condition. ◇

## V. Influence Matrix for PIR Aggregates

Once established the stability of the dynamical network, we can perform a structural steady-state analysis, whose outcome is encoded in the so-called influence matrix.

We consider the steady-state behaviour of model (5) subject to a perturbing input $u$,

$$y(s) = \Phi(\Sigma, s)y(s) + \Delta u(s), \tag{12}$$

where $\Phi$ is defined as in (6), $\Delta$ is a column vector with nonnegative entries and $u$ is a step input: $u(t) = \bar{u}$. The steady-state equation

$$\bar{y} = [I - \Phi(\Sigma, 0)]^{-1}\Delta\bar{u} \tag{13}$$

represents the steady-state effect of the persistent input $u$ on the output $y$ [12]. In the context of aggregate systems, matrix

$$M = \text{sign}[I - \Phi(\Sigma, 0)]^{-1}$$

is the *structural influence matrix*, whose entry $M_{ij}$ represents the structural steady-state effect of an input applied to the $j$th subsystem on the output of the $i$th subsystem, for any possible choice of the system parameters.

Assuming that the entries of the interaction matrix $\Sigma$ are bounded as

$$\eta_{ij} \le \sigma_{ij} \le w_{ij}, \tag{14}$$

where $\eta_{ij}$ and $w_{ij}$ are given numbers (possibly $-\infty$ or $+\infty$, respectively), we have that

- $M_{ij}$ = '+' if $\text{sign}([I - \Phi(\Sigma, 0)]^{-1})_{ij} = 1$ for all possible $\sigma_{ij}$ as in (14);
- $M_{ij}$ = '−' if $\text{sign}([I - \Phi(\Sigma, 0)]^{-1})_{ij} = -1$ for all possible $\sigma_{ij}$ as in (14);
- $M_{ij}$ = '0' if $\text{sign}([I - \Phi(\Sigma, 0)]^{-1})_{ij} = 0$ for all possible $\sigma_{ij}$ as in (14);
- $M_{ij}$ = '?' if $\text{sign}([I - \Phi(\Sigma, 0)]^{-1})_{ij}$ can vary depending on $\sigma_{ij}$ as in (14).

Under stability assumptions, the entries of the influence matrix are multi-affine functions of the variables $\sigma_{ij}$. Therefore, we can propose a vertex criterion that extends the criterion proposed in [12] for systems admitting the so-called BDC-decomposition $\dot{x} = BDCx$ [6], [10], [12].

Define as vert$[\eta, \omega]$ the set of all the vertices of the box defined by (14), namely, all the points with $\sigma_{ij}$ taken on the extrema:

$$\sigma_{ij} \in \{\eta_{ij}, w_{ij}\}.$$

*Theorem 3:* Assume that the overall system (12), with $\Phi$ defined as in (6), is Hurwitz for all $\sigma_{ij}$ as in (14). Let adj$[A] = A^{-1}\det[A]$ denote the adjoint matrix of $A$. Then

- $M_{ij}$ = '+' if sign(adj$[I - \Sigma]$)$_{ij} = 1$ on vert$[\eta, \omega]$;
- $M_{ij}$ = '−' if sign(adj$[I - \Sigma]$)$_{ij} = -1$ on vert$[\eta, \omega]$ ;
- $M_{ij}$ = '0' if sign(adj$[I - \Sigma]$)$_{ij} = 0$ on vert$[\eta, \omega]$;
- $M_{ij}$ = '?' if sign(adj$[I - \Sigma]$)$_{ij}$ can have both positive and negative sign on vert$[\eta, \omega]$.

□

*Proof:* Let us restrict to the case of $w_{ij} < \infty$ and $\eta_{ij} > -\infty$. If we assume stability, then

$$\det[I - \Phi(\Sigma, 0)] > 0 \tag{15}$$

for all $\sigma_{ij}$ as in (14), because any change of sign would imply that $[I - \Phi(\Sigma, s)] = 0$ for $s = 0$, hence the system would not be Hurwitz stable. Then, since (15) holds for all $\sigma_{ij}$ as in (14), the entries of the adjoint matrix are multi-affine functions of the interaction weights $\sigma_{ij}$, hence the minimum and the maximum value are achieved on the vertices of the box and the proof goes along the lines in [12].

A proof can be provided also for the case in which, possibly, $w_{ij} = +\infty$ or $\eta_{ij} = -\infty$, by considering "the infinite vertices". This situation is involved and is not consider here for space limits. ■

*Remark 3:* We can check if $\det[I - \Phi(\Sigma, 0)] > 0$ for all $\sigma_{ij}$ as in (14) by testing all the vertices in vert$[\eta, \omega]$. In fact, also the determinant in (15) is a multi-affine function of the variables $\sigma_{ij}$. ◇

*Remark 4:* In some cases, the robust stability test may fail and we may have instability for some choice of the parameter values. Still, computing the influence matrix can reveal some sign-definite steady-state effects: hence, the influence is structurally positive, negative, or zero for all the subsets of parameters that ensure stability of the system. ◇

## VI. Examples

*Example 5:* Reconsider Example 2, with matrix $\Sigma$ as in (7). The general stability condition of Theorem 1 requires Hurwitz stability of matrix $-I + \Omega$, whose expression is in (11). The corresponding characteristic polynomial is

$$p(s) = (s+1)^4 - w_{12}w_{21}(s+1)^2 - w_{14}w_{21}w_{32}w_{43}.$$

Since matrix $-I + \Omega$ is Metzler, a *necessary and sufficient* condition for Hurwitz stability is that the coefficients of the polynomial $p(s)$ are positive. This leads to the condition

$$w_{14}w_{21}w_{32}w_{43} + w_{12}w_{21} < 1.$$

If this condition is not verified, then the system is not structurally stable; yet, it can be stable for some choice of the parameter values and transfer functions. ◇

*Example 6:* Consider an aggregate of four PIR subsystems whose interaction matrix has sign pattern

$$S = \begin{bmatrix} 0 & + & 0 & + \\ - & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & - & 0 \end{bmatrix}. \tag{16}$$

Without upper or lower bounds for $\sigma_{ij}$, the influence matrix is

$$M = \begin{bmatrix} + & + & + & + \\ - & + & - & - \\ - & + & + & - \\ - & + & + & + \end{bmatrix}. \tag{17}$$

This means that all possible influences of the input of subsystem $j$ on the output of subsystem $i$ are structurally sign definite, whenever the parameter values ensure stability. Clearly, stability is not ensured unless the magnitude of the weights $\sigma_{ij}$ is bounded. ⋄

*Example 7:* Given an aggregate of four PIR subsystems whose interaction matrix has sign pattern

$$S = \begin{bmatrix} 0 & + & 0 & - \\ + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \end{bmatrix}, \tag{18}$$

without upper or lower bounds on $\sigma_{ij}$, the resulting influence matrix is

$$M = \begin{bmatrix} + & ? & - & - \\ + & + & - & - \\ + & + & ? & - \\ + & + & ? & ? \end{bmatrix}. \tag{19}$$

The sign pattern is not entirely sign definite. The '?' sign on the diagonal might seem strange: it tells us that a permanent input applied to subsystem $i$ may lead to a steady-state variation of the output of the same subsystem having opposite sign. This simply means that a non-minimum phase behaviour can occur for some values of the parameters. ⋄

## VII. Concluding Discussion

A very natural and general system decomposition of a dynamical network, in particular when biological systems and biochemical systems are considered, can be achieved by seeing the system as an aggregate of Positive Impulse Response (PIR) subsystems [9]. This generalises the decomposition into aggregates of input-output monotone systems [3], [8], [23], [24], a very successful approach in the study of biological systems.

In particular, in this paper we have considered arbitrary interconnections of subsystems characterised by a positive impulse response, which can be associated with a graph whose nodes represent PIR subsystems and whose weighted arcs represent the interactions among subsystems, weighted by their strength.

For this general class of systems, we have proposed new parameter-free results concerning a structural stability analysis and the determination of the influence matrix, expressing the structural steady-state effect of a persistent input applied to one of the subsystems to the output of another subsystem. Precisely, we have shown that structural stability of the overall interconnected system is ensured, for any choice of the PIR subsystems and of the interaction weights, if a certain Metzler matrix depending on the interaction weights is Hurwitz. As for the structural steady-state analysis whose result is summarised by the influence matrix, we have shown that a vertex result proposed in [12] can be suitably adapted to the case of aggregates of subsystems associated with stable transfer functions.

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