

A saturated strategy robustly ensures stability of the cooperative equilibrium for Prisoner's dilemma

Giulia Giordano¹, Dario Bauso², and Franco Blanchini³

Abstract—We study diffusion of cooperation in a two-population game in continuous time. At each instant, the game involves two random individuals, one from each population. The game has the structure of a Prisoner's dilemma where each player can choose either to cooperate (c) or to defect (d), and is reframed within the field of approachability in two-player repeated game with vector payoffs. We turn the game into a dynamical system, which is positive, and propose a saturated strategy that ensures local asymptotic stability of the equilibrium (c, c) for any possible choice of the payoff matrix. We show that there exists a rectangle, in the space of payoffs, which is positively invariant for the system. We also prove that there exists a region in the space of payoffs for which the equilibrium solution (d, d) is an attractor, while all of the trajectories originating outside that region, but still in the positive quadrant, are ultimately bounded in the rectangle and, under suitable assumptions, converge to the solution (c, c) .

I. INTRODUCTION

Two large populations of individuals play a game in continuous time. At each instant, a random individual of the first population engages in play with a random opponent extracted from the second population. The resulting payoff, which depends on the action profiles of both players, is a vector. The game is a Prisoner's dilemma [25], [16]: each player can choose either to cooperate (c) or to defect (d). The defection of a single player is most beneficial for the defecting player and most harmful for the other player, while for both players mutual cooperation is preferable to mutual defection. We reframe the problem within the field of approachability in two-player repeated game with vector payoffs [8], [7].

In a two-player repeated game with vector payoffs, a set of payoffs is approachable [10] if the row player has a strategy such that, for any strategy used by her opponent, her average payoff uniformly approaches the set with

probability 1. The notion of *approachability* is due to [10]: Blackwell's Theorem, giving conditions for a set to be approachable, is often used to prove convergence in different application domains, *e.g.*, allocation processes in coalitional games [17], regret minimization [19], [15] and weak approachability [28]. A similar concept can be found in adaptive learning and evolutionary games [14], [24]. The original discrete-time formulation of approachability has been adapted to continuous-time repeated games in [15], which also highlights the connection with Lyapunov theory. An extension to infinite-dimensional spaces is due to [18]. Approachability shares striking similarities with differential game theory and, as such, can be studied using differential calculus and stability theory [22], [26]. Approachability and differential inclusions [2] are studied in [22], where it is highlighted that Blackwell's theorem is a generalization of von Neumann's minmax theorem [29]. [26] proposes a set-valued analytical perspective [1], [3]; approachable and discriminating sets can be reframed within the context of set invariance theory [11]. A core concept in approachability is that of nonanticipative strategies, similar to those in differential games [5], [12], [26], [23], [27]; classical feedback strategies in differential games are special nonanticipative strategies. *Excludability* of sets is a complementary notion to approachability [20] and another concept related to approachability is *attainability* [6], [21], useful in application domains such as transportation, distribution and production networks.

In this paper, we set up the approachability problem for the two-player repeated game with vector payoffs in a system-theoretical framework, turning the game into a positive dynamical system and showing that it can be reviewed as a population game. In Smale's *good strategies* for Prisoner's dilemma [25] and subsequent developments [9], [8], [7], the decision of each player is based on the knowledge of the whole current average payoff vector. Here, we propose a novel saturated strategy where each player's decision is based on the exclusive knowledge of her/his own current average payoff and of the diagonal entries of her/his own payoff matrix: information about the other player is not required, which makes this strategy well suited also for games with incomplete information [4]. The main contributions can

¹ Department of Automatic Control and LCCC Linnaeus Center, Lund University, Box 118, SE 221 00 Lund, Sweden. giulia.giordano@control.lth.se.

² Department of Automatic Control and Systems Engineering, The University of Sheffield, Mappin Street, Sheffield, S1 3JD, United Kingdom; and also DICGIM, Università di Palermo, V.le delle Scienze, 90128, Palermo, Italy. d.bauso@sheffield.ac.uk.

³ Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università degli Studi di Udine, Via delle Scienze 206, 33100 Udine, Italy. blanchini@uniud.it.

be summarized as follows:

- the saturated strategy, for both players, ensures that the equilibrium (c, c) is locally asymptotically stable for any choice of the payoff matrix;
- there exists a rectangle \mathcal{R} , in the space of payoffs, which is a positively invariant set for the system;
- there exists a region \mathcal{D} , in the space of payoffs, for which the equilibrium solution (d, d) is an attractor, namely, all of the trajectories originating in the region converge to the equilibrium;
- all of the trajectories originating outside region \mathcal{D} , but still in the positive quadrant, are ultimately bounded in the rectangle \mathcal{R} : if the equilibrium (d, d) is unstable, they all converge to the equilibrium (c, c) ;
- under suitable assumptions on the payoff matrix values, the solution (d, d) can be rendered unstable by a proper choice of the saturated strategy.

Numerical simulations illustrate the evolution of the game with the proposed saturated strategy.

II. PROBLEM FORMULATION AND MOTIVATION

We consider a Prisoner's dilemma with two players, each striving to maximise its payoff. Each player can choose either to cooperate or to defect; depending on the players' choice, the average payoff vector (x, y) evolves according to the game payoff matrix

$$\begin{bmatrix} (\alpha_1, \alpha_2) & (\gamma_1, \gamma_2) \\ (\beta_1, \beta_2) & (\delta_1, \delta_2) \end{bmatrix}, \quad (1)$$

whose entries represent the payoff vectors when both players cooperate (α), player 1 only defects (β), player 2 only defects (γ) and both players defect (δ). Player 1, whose average payoff is x , chooses the row, while player 2, whose average payoff is y , chooses the column. In the usual formulation, the defection of a single player is beneficial to the highest degree for the defecting player and harmful to the highest degree for the other player, while for both players mutual cooperation is more advantageous than mutual defection.

Assumption 1: In the payoff matrix (1),

$$\begin{cases} \beta_1 > \alpha_1 > \delta_1 > \gamma_1, \\ \gamma_2 > \alpha_2 > \delta_2 > \beta_2. \end{cases} \quad (2)$$

Fig. 1a shows the points corresponding to the payoff vectors in the outcome plane (x, y) : $A = (\alpha_1, \alpha_2)$, $B = (\beta_1, \beta_2)$, $C = (\gamma_1, \gamma_2)$ and $D = (\delta_1, \delta_2)$.

We assume that the two players adopt a mixed strategy

$$u = u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \quad v = v(y) = \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix},$$

such that $u_1 + u_2 = 1$, with $u_1, u_2 \geq 0$, and $v_1 + v_2 = 1$, with $v_1, v_2 \geq 0$. Then, defining the matrices

$$F = \begin{bmatrix} \alpha_1 & \gamma_1 \\ \beta_1 & \delta_1 \end{bmatrix}, \quad G = \begin{bmatrix} \alpha_2 & \gamma_2 \\ \beta_2 & \delta_2 \end{bmatrix},$$

the evolution of the average payoff in the repeated game (or, as will be discussed in Section II-A, of the average payoff over the population) is described by the system

$$\begin{cases} \dot{x}(t) = -x(t) + u(x(t))^\top Fv(y(t)), \\ \dot{y}(t) = -y(t) + u(x(t))^\top Gv(y(t)). \end{cases} \quad (3)$$

Given $h, k \in \mathbb{R}$ such that $h < k$, we define the *saturation function* as

$$\text{sat}_{[h k]}f(x) = \begin{cases} k & \text{if } f(x) \geq k, \\ f(x) & \text{if } h < f(x) < k, \\ h & \text{if } f(x) \leq h, \end{cases} \quad (4)$$

and we consider the *saturated strategy*

$$\begin{cases} u_1(x) = 1 - \sigma_1(x), \\ u_2(x) = \sigma_1(x), \\ v_1(y) = 1 - \sigma_2(y), \\ v_2(y) = \sigma_2(y), \end{cases} \quad (5)$$

where

$$\begin{cases} \sigma_1(x) = \left[\text{sat}_{[0 1]} \left(\frac{\alpha_1 - x}{\alpha_1 - \delta_1} \right) \right]^p, \\ \sigma_2(y) = \left[\text{sat}_{[0 1]} \left(\frac{\alpha_2 - y}{\alpha_2 - \delta_2} \right) \right]^p, \end{cases} \quad (6)$$

for $p \in \mathbb{N}$, $p \geq 1$. Function $\sigma_1(x)$ is illustrated in Fig. 1c for various values of p : the larger is p , the steeper is the saturated function in the interval $[\delta_1, \alpha_1]$.

For each player, the proposed strategy is exclusively based on her/his own current average payoff and on the diagonal entries of her/his own payoff matrix and information about the other player is not required.

Adopting the strategy (5)–(6), system (3) becomes

$$\begin{cases} \dot{x} = -x + \alpha_1 + (\beta_1 - \alpha_1)\sigma_1(x) + (\gamma_1 - \alpha_1)\sigma_2(y) \\ \quad + (\alpha_1 + \delta_1 - \beta_1 - \gamma_1)\sigma_1(x)\sigma_2(y), \\ \dot{y} = -y + \alpha_2 + (\beta_2 - \alpha_2)\sigma_1(x) + (\gamma_2 - \alpha_2)\sigma_2(y) \\ \quad + (\alpha_2 + \delta_2 - \beta_2 - \gamma_2)\sigma_1(x)\sigma_2(y). \end{cases} \quad (7)$$

A. A Population-Game Perspective

Equation (3) is in the same spirit as in Hart and Mas-Colell's paper [15] on continuous-time approachability. To see this, consider the time-average expected (over opponent's play) payoff defined as

$$\Gamma(s) = \frac{1}{s} \int_0^s \begin{bmatrix} u^\top Fv \\ u^\top Gv \end{bmatrix} d\tau \in \mathbb{R}^2.$$

If we rescale the time window using $s = e^t$, take $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Gamma(e^t)$ and differentiate with respect to t , we obtain the differential equation (3). Note that, after rescaling the time window, we have

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \int_0^1 \begin{bmatrix} u^\top Fv \\ u^\top Gv \end{bmatrix} d\tau \in \mathbb{R}^2.$$

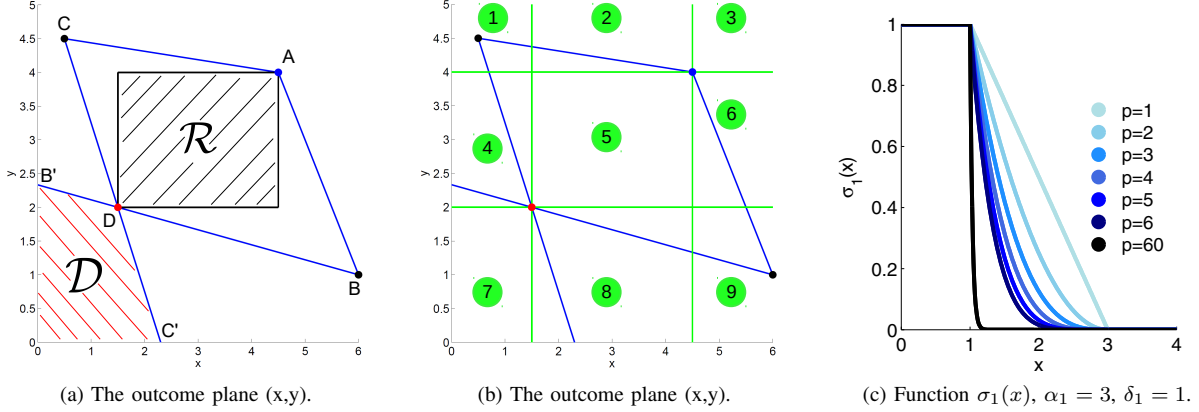


Fig. 1: The outcome plane (x,y): (a) with the indication of sets \mathcal{D} and \mathcal{R} ; (b) divided in nine regions. (c) Plot of function $\sigma_1(x)$, with $\alpha_1 = 3$ and $\delta_1 = 1$, for various values of p .

Adopting a *population-game dynamics* perspective, the state $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^2$ represents the current average payoff over the population.

III. MAIN RESULTS

We begin by showing that both the cooperative point A and the non-cooperative point D are equilibria for the dynamical system (7), and that A is always locally asymptotically stable if a saturated strategy with $p > 1$ is chosen, regardless of parameter values.

Proposition 1: For any $p \geq 1$, both $A = (\alpha_1, \alpha_2)$ and $D = (\delta_1, \delta_2)$ are equilibria for system (7). Furthermore, for $p > 1$, $A = (\alpha_1, \alpha_2)$ is locally asymptotically stable for any choice of the values in the payoff matrix (1).

Proof: If $(x, y) = (\alpha_1, \alpha_2)$, then $\sigma_1(x) = \sigma_2(y) = 0$, while, if $(x, y) = (\delta_1, \delta_2)$, then $\sigma_1(x) = \sigma_2(y) = 1$; in both cases, as can be computed by substitution in system (7), $\dot{x} = \dot{y} = 0$, hence both of the points are equilibria for the system.

The system Jacobian matrix in the non-saturated region is shown in Table I. When computed at the cooperative equilibrium, for $p > 1$, it becomes $J_{(\alpha_1, \alpha_2)} = -I$, where I is the identity (and analogously for the saturated regions 2, 3 and 6 in Fig. 1b). Hence, the equilibrium $A = (\alpha_1, \alpha_2)$ is locally asymptotically stable, for any choice of the payoff matrix values. ■

The equilibrium A is not necessarily stable if $p = 1$, while the equilibrium D can be either stable or unstable, depending on the payoff matrix values and on p .

Example 1: Consider system (7) with payoff matrix

$$\begin{bmatrix} (\alpha_1, \alpha_2) & (\gamma_1, \gamma_2) \\ (\beta_1, \beta_2) & (\delta_1, \delta_2) \end{bmatrix} = \begin{bmatrix} (4.5, 4) & (0.5, 4.5) \\ (6, 1) & (1.5, 2) \end{bmatrix}. \quad (8)$$

If $p = 1$, the equilibrium $A = (\alpha_1, \alpha_2)$ is unstable, while the equilibrium $D = (\delta_1, \delta_2)$ is asymptotically stable.

To state our main theorem, we need to consider two sets in the outcome plane: the quadrilateral

$$\begin{aligned} \mathcal{D} &= \{(x, y) : x \geq 0, y \geq 0, \\ &(\beta_1 - \delta_1)y + (\delta_2 - \beta_2)x \leq \beta_1\delta_2 - \beta_2\delta_1, \\ &(\gamma_1 - \delta_1)y + (\delta_2 - \gamma_2)x \geq \gamma_1\delta_2 - \gamma_2\delta_1\} \end{aligned} \quad (9)$$

and the rectangle corresponding to the non-saturated region,

$$\mathcal{R} = \{(x, y) : \delta_1 \leq x \leq \alpha_1 \text{ and } \delta_2 \leq y \leq \alpha_2\}, \quad (10)$$

both shown in Fig. 1a.

Theorem 1: Given system (7) under Assumption 1, the following statements hold.

- System (7) is positive.¹
- The rectangle \mathcal{R} is a positively invariant set for the system.
- All of the trajectories originating in \mathcal{D} converge to the equilibrium $D = (\delta_1, \delta_2)$.
- All of the trajectories originating in $\mathbb{R}_+^2 \setminus \mathcal{D}$ are ultimately bounded in \mathcal{R} .

Proof: The lines $x = \delta_1$, $y = \delta_2$, $x = \alpha_1$ and $y = \alpha_2$ divide the positive orthant $\mathbb{R}_+^2 = \{(x, y) : x \geq 0 \text{ and } y \geq 0\}$ in nine regions, as shown in Fig. 1b. According to the values of the saturation functions inside each region, we can compute the derivatives \dot{x} and \dot{y} .

- Region 1: $x < \delta_1$ and $y > \alpha_2$, hence $\sigma_1(x) = 1$ and $\sigma_2(y) = 0$, $\dot{x} = -x + \beta_1 > 0$ and $\dot{y} = -y + \beta_2 < 0$.
- Region 2: $\delta_1 \leq x \leq \alpha_1$ and $y > \alpha_2$, hence $\sigma_2(y) = 0$, $\dot{x} = -x + \alpha_1 + (\beta_1 - \alpha_1)\sigma_1(x) \geq 0$ (strictly positive if $x \neq \alpha_1$) and $\dot{y} = -y + \alpha_2 + (\beta_2 - \alpha_2)\sigma_1(x) < 0$.
- Region 3: $x > \alpha_1$ and $y > \alpha_2$, hence $\sigma_1(x) = \sigma_2(y) = 0$, $\dot{x} = -x + \alpha_1 < 0$ and $\dot{y} = -y + \alpha_2 < 0$.
- Region 4: $x < \delta_1$ and $\delta_2 \leq y \leq \alpha_2$, hence $\sigma_1(x) = 1$, $\dot{x} = -x + \beta_1 + (\delta_1 - \beta_1)\sigma_2(y) \geq (\beta_1 - \delta_1)[1 - \sigma_2(y)] > 0$

¹A system is *positive* if the positive orthant is a positively invariant set for the system: the state variables are always positive in value.

$$J_{(x,y)} = \begin{bmatrix} -1 - p(\alpha_1 - x)^{p-1} \left[\frac{\beta_1 - \alpha_1}{(\alpha_1 - \delta_1)^p} + \frac{(\alpha_1 + \delta_1 - \beta_1 - \gamma_1)(\alpha_2 - y)^p}{(\alpha_1 - \delta_1)^p(\alpha_2 - \delta_2)^p} \right] & -p(\alpha_2 - y)^{p-1} \left[\frac{\gamma_1 - \alpha_1}{(\alpha_2 - \delta_2)^p} + \frac{(\alpha_1 + \delta_1 - \beta_1 - \gamma_1)(\alpha_1 - x)^p}{(\alpha_1 - \delta_1)^p(\alpha_2 - \delta_2)^p} \right] \\ -p(\alpha_1 - x)^{p-1} \left[\frac{\beta_2 - \alpha_2}{(\alpha_1 - \delta_1)^p} + \frac{(\alpha_2 + \delta_2 - \beta_2 - \gamma_2)(\alpha_2 - y)^p}{(\alpha_1 - \delta_1)^p(\alpha_2 - \delta_2)^p} \right] & -1 - p(\alpha_2 - y)^{p-1} \left[\frac{\gamma_2 - \alpha_2}{(\alpha_2 - \delta_2)^p} + \frac{(\alpha_2 + \delta_2 - \beta_2 - \gamma_2)(\alpha_1 - x)^p}{(\alpha_1 - \delta_1)^p(\alpha_2 - \delta_2)^p} \right] \end{bmatrix}$$

TABLE I: Jacobian of system (7) for $\delta_1 \leq x \leq \alpha_1$ and $\delta_2 \leq y \leq \alpha_2$.

and $\dot{y} = -y + \beta_2 + (\delta_2 - \beta_2)\sigma_2(y) \leq (\beta_2 - \delta_2)[1 - \sigma_2(y)] \leq 0$ (strictly negative if $y \neq \delta_2$).

- **Region 5**, namely region \mathcal{R} in (10), is positively invariant: in fact, if $x = \delta_1$, $\sigma_1(x) = 1$ and $\dot{x} = (\beta_1 - \delta_1)[1 - \sigma_2(y)] \geq 0$; if $x = \alpha_1$, $\sigma_1(x) = 0$ and $\dot{x} = (\gamma_1 - \alpha_1)\sigma_2(y) \leq 0$; if $y = \delta_2$, $\sigma_2(y) = 1$ and $\dot{y} = (\gamma_2 - \delta_2)[1 - \sigma_1(x)] \geq 0$; if $y = \alpha_2$, $\sigma_2(y) = 0$ and $\dot{y} = (\beta_2 - \alpha_2)\sigma_1(x) \leq 0$. This proves statement (b).

- **Region 6**: $x > \alpha_1$ and $\delta_2 \leq y \leq \alpha_2$, hence $\sigma_1(x) = 0$, $\dot{x} = -x + \alpha_1 + (\gamma_1 - \alpha_1)\sigma_2(y) < 0$ and $\dot{y} = -y + \alpha_2 + (\gamma_2 - \alpha_2)\sigma_2(y) \geq 0$ (null only if $y = \alpha_2$).

- **Region 7**: $x < \delta_1$ and $y < \delta_2$, hence $\sigma_1(x) = \sigma_2(y) = 1$, $\dot{x} = -x + \delta_1 > 0$ and $\dot{y} = -y + \delta_2 > 0$.

- **Region 8**: $\delta_1 \leq x \leq \alpha_1$ and $y < \delta_2$, hence $\sigma_2(y) = 1$, $\dot{x} = -x + \gamma_1 + (\delta_1 - \gamma_1)\sigma_1(x) \leq (\gamma_1 - \delta_1)[1 - \sigma_1(x)] \leq 0$ (strictly negative if $x \neq \delta_1$) and $\dot{y} = -y + \gamma_2 + (\delta_2 - \gamma_2)\sigma_1(x) > (\gamma_2 - \delta_2)[1 - \sigma_1(x)] \geq 0$.

- **Region 9**: $x > \alpha_1$ and $y < \delta_2$, hence $\sigma_1(x) = 0$ and $\sigma_2(y) = 1$, $\dot{x} = -x + \gamma_1 < 0$ and $\dot{y} = -y + \gamma_2 > 0$.

Statement (a) follows from the fact that, for any point of the regions 1, 4 and 7, including $x = 0$, $\dot{x} > 0$, while for any point of the regions 7, 8 and 9, including $y = 0$, $\dot{y} > 0$, hence the positive orthant is positively invariant.

Consider now the segments $B'D$ and DC' in Fig. 1a: segment $B'D$ lies on the line $(\beta_1 - \delta_1)y + (\delta_2 - \beta_2)x = \beta_1\delta_2 - \beta_2\delta_1$, while segment DC' lies on the line $(\gamma_1 - \delta_1)y + (\delta_2 - \gamma_2)x = \gamma_1\delta_2 - \gamma_2\delta_1$. If we compute the normal component of the derivative, for all points belonging to these segments, and we recall that $\sigma_1(x) = 1$ on $B'D$ and $\sigma_2(y) = 1$ on DC' , we obtain $(\beta_1 - \delta_1)\dot{y} + (\delta_2 - \beta_2)\dot{x} = 0$ and $(\gamma_1 - \delta_1)\dot{y} + (\delta_2 - \gamma_2)\dot{x} = 0$. Hence, no trajectory can cross these lines (actually, the two segments are invariant sets). As a consequence, the trajectories originating in \mathcal{D} are bounded in \mathcal{D} and, due to the sign of the derivatives in regions 4, 7 and 8, converge to the equilibrium D for large enough time, thus proving statement (c).

The signs of the derivatives computed above also show that any trajectory starting in $\mathbb{R}_+^2 \setminus \mathcal{D}$ (which is bounded in $\mathbb{R}_+^2 \setminus \mathcal{D}$) converges to \mathcal{R} for large enough time, hence proving statement (d). ■

The proof of Theorem 1 entails the following result.

Corollary 1: System (7) does not admit equilibrium points outside the rectangle \mathcal{R} .

Does the proposed saturated strategy, with p large enough, ensure that all trajectories originating in $\mathbb{R}_+^2 \setminus \mathcal{D}$ converge to the equilibrium $A = (\alpha_1, \alpha_2)$, when the equilibrium D is unstable? To investigate this problem,

we just need to consider all of the trajectories originating in $\mathcal{R} \setminus D$ (since all trajectories in $\mathbb{R}_+^2 \setminus \mathcal{D}$ are ultimately bounded in \mathcal{R} and $D = (\delta_1, \delta_2)$ is an equilibrium point).

The system Jacobian computed at $D = (\delta_1, \delta_2)$ is

$$J_{(\delta_1, \delta_2)} = \begin{bmatrix} -1 - p \frac{\delta_1 - \gamma_1}{\alpha_1 - \delta_1} & p \frac{\beta_1 - \delta_1}{\alpha_2 - \delta_2} \\ p \frac{\gamma_2 - \delta_2}{\alpha_1 - \delta_1} & -1 - p \frac{\delta_2 - \beta_2}{\alpha_2 - \delta_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + p \begin{bmatrix} -\frac{\delta_1 - \gamma_1}{\alpha_1 - \delta_1} & \frac{\beta_1 - \delta_1}{\alpha_2 - \delta_2} \\ \frac{\gamma_2 - \delta_2}{\alpha_1 - \delta_1} & -\frac{\delta_2 - \beta_2}{\alpha_2 - \delta_2} \end{bmatrix} = -I + p\Theta.$$

Both $J_{(\delta_1, \delta_2)}$ and Θ are irreducible Metzler matrices (their off-diagonal entries are nonnegative) with negative diagonal entries. It is worth recalling that a Metzler matrix has a real dominant eigenvalue, associated with a positive eigenvector (the Frobenius eigenvector) [13].

Lemma 1: There exists a finite $p \geq 1$ for which the equilibrium D of system (7) is exponentially unstable if and only if matrix Θ has a positive eigenvalue.

Proof: It is immediate, since, denoting by θ_i (with $i = 1, 2$) the eigenvalues of Θ , which are real, the eigenvalues of $J_{(\delta_1, \delta_2)}$ are $\lambda_i = -1 + p\theta_i$. ■

Then, we can state the following.

Theorem 2: There exists a finite $p \geq 1$ such that all of the trajectories of system (7) originating in $\mathcal{R} \setminus (\delta_1, \delta_2)$ converge to the equilibrium A if and only if matrix Θ has a positive eigenvalue.

Proof: Necessity. By contradiction, if matrix Θ does not have positive eigenvalues, then the equilibrium D is locally asymptotically stable and for any p there exists a neighborhood \mathcal{N} of D , having a nonempty intersection with $\mathcal{R} \setminus (\delta_1, \delta_2)$, such that the trajectories originating in \mathcal{N} converge to the equilibrium D , and not to the equilibrium A .

Sufficiency. If matrix Θ has a positive eigenvalue, then the equilibrium D is unstable for p large enough: the corresponding Jacobian $J_{(\delta_1, \delta_2)}$ has a real positive eigenvalue associated with a positive Frobenius eigenvector. Consider, for simplicity, the new variables $z = \frac{\alpha_1 - x}{\alpha_1 - \delta_1}$ and $w = \frac{\alpha_2 - y}{\alpha_2 - \delta_2}$. In \mathcal{R} , both z and w take values between 0 and 1. This change of variables, applied to system (7), gives the system

$$\begin{cases} \dot{z} = -z - \mu_1 z^p + \nu_1 w^p + (1 + \mu_1 - \nu_1)z^p w^p, \\ \dot{w} = -w + \mu_2 z^p - \nu_2 w^p + (1 - \mu_2 + \nu_2)z^p w^p, \end{cases} \quad (11)$$

where $\mu_1 = \frac{\beta_1 - \alpha_1}{\alpha_1 - \delta_1}$, $\nu_1 = -\frac{\gamma_1 - \alpha_1}{\alpha_1 - \delta_1}$, $\mu_2 = -\frac{\beta_2 - \alpha_2}{\alpha_2 - \delta_2}$, $\nu_2 = \frac{\gamma_2 - \alpha_2}{\alpha_2 - \delta_2}$ are positive values. The equilibrium points are transformed as $A' = (0, 0)$ and $D' = (1, 1)$ and the

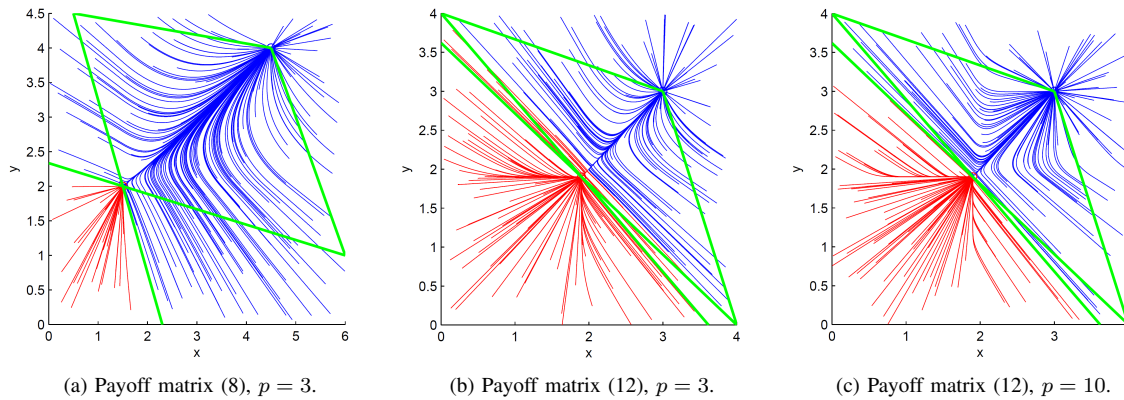


Fig. 3: System evolution with the saturated strategy: trajectories with random initial conditions in the positive quadrant.

and with $p = 3$, some of the trajectories originating in $\mathbb{R}_+^2 \setminus \mathcal{D}$ converge to D , which is a stable equilibrium. A new unstable equilibrium appears in between, which is the turning point between trajectories converging to A and trajectories converging to D . However, the choice $p = 10$ destabilizes D : then, all of the trajectories originating in $\mathbb{R}_+^2 \setminus \mathcal{D}$ converge to A , as shown in Fig. 3c, consistently with our results.

V. CONCLUSIONS

We have shown that a saturated strategy robustly ensures stability of the cooperative equilibrium of a two-population game having the structure of a Prisoner's dilemma. Future directions involve the analysis of the n -dimensional case and the extension of the analysis to a larger set of games including the *coordination game*.

ACKNOWLEDGEMENTS

G.G. acknowledges support from the Swedish Research Council through the LCCC Linnaeus Center and the eLLIIT Excellence Center at Lund University.

REFERENCES

- [1] J. P. Aubin. *Viability Theory*. Birkhäuser, 1991.
- [2] J. P. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer, 1991.
- [3] J. P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, 1990.
- [4] Robert J. Aumann and Michael B. Maschler. *Repeated Games with Incomplete Information*. MIT Press, 1995.
- [5] F. Bagagiolo and D. Bauso. Objective function design for robust optimality of linear control under state-constraints and uncertainty. *ESAIM: Control, Optimisation and Calculus of Variations*, 17:155–177, 2011.
- [6] D. Bauso, E. Lehrer, and E. Solan. Attainability in repeated games with vector payoffs. *arXiv:1201.6054v2*, 2012.
- [7] K. Behrstock, M. Benaïm and M. W. Hirsch. Smale strategies for network prisoner's dilemma games. *J. Dynamics and Games* 2:141–155, 2015.
- [8] M. Benaïm, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions. *SIAM Journal on Control and Optimization*, 44(1):338–348, 2005.
- [9] M. Benaïm and M. W. Hirsch. Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations*, 8:141–176, 1996.
- [10] D. Blackwell. An analog of the minimax theorem for vector payoffs. *Pacific J. Math.*, 6(1):1–8, 1956.
- [11] F. Blanchini. Set invariance in control: a survey. *Automatica*, 35(11):1747–1768, 1999.
- [12] N. J. Elliot and N.J. Kalton. The existence of value in differential games of pursuit and evasion. *J. Differential Equations*, 12:504–523, 1972.
- [13] L. Farina and S. Rinaldi. *Positive Linear Systems; Theory and Applications*. John Wiley & Sons, 2000.
- [14] S. Hart and A. Mas-Colell. A general class of adaptive strategies. *Journal of Economic Theory*, 98:26–54, 2001.
- [15] S. Hart and A. Mas-Colell. Regret-based continuous-time dynamics. *Games and Economic Behavior*, 45:375–394, 2003.
- [16] S. Huck, H.-T. Normann, J. Oechssler. GLAD: a simple adaptive strategy that yields cooperation in dilemma games. *Physica D* 200:133–138, 2005.
- [17] E. Lehrer. Allocation processes in cooperative games. *International Journal of game Theory*, 31:341–351, 2002.
- [18] E. Lehrer. Approachability in infinite dimensional spaces. *International Journal of game Theory*, 31(2):253–268, 2002.
- [19] E. Lehrer. A wide range no-regret theorem. *Games and Economic Behavior*, 42, 2003.
- [20] E. Lehrer and E. Solan. Excludability and bounded computational capacity strategies. *Mathematics of Operations Research*, 31(3):637–648, 2006.
- [21] E. Lehrer, E. Solan, and D. Bauso. Repeated games over networks with vector payoffs: the notion of attainability. In *Proceedings of the NetGCoop 2011*, Paris, France, October 2011.
- [22] E. Lehrer and S. Sorin. Minimax via differential inclusion. *Convex Analysis*, 14(2):271–273, 2007.
- [23] E. Roxin. The axiomatic approach in differential games. *J. Optim. Theory Appl.*, 3:153–163, 1969.
- [24] W.H. Sandholm. *Population Games and Evolutionary Dynamics*. MIT Press, 2010.
- [25] S. Smale. The prisoner's dilemma and dynamical systems associated to non-cooperative games. *Econometrica*, 48:1617–1634, 1980.
- [26] A. S. Soulaïmani, M. Quincampoix, and S. Sorin. Approachability theory, discriminating domain and differential games. *SIAM Journal of Control and Optimization*, 48(4):2461–2479, 2009.
- [27] P. Varaiya. The existence of solution to a differential game. *SIAM Journal of Control and Optimization*, 5:153–162, 1967.
- [28] N. Vieille. Weak approachability. *Mathematics of Operations Research*, 17:781–791, 1992.
- [29] John von Neumann. Zur Theorie der Gesellschaftsspiele. *Math. Annalen*, 100:295–320, 1928.